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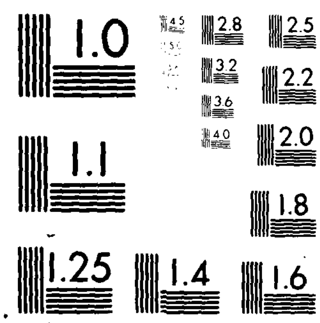

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LEVEL II

REDESIGN OF THE STABILIZED PITCH CONTROL SYSTEM OF A SEMI-ACTIVE TERMINAL HOMING MISSILE SYSTEM

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20 April 1979

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A high-order stabilization filter was formerly designed to stabilize an unstable pitch control system of a terminal homing missile system. In this report, a new dominant-data matching method is presented to redesign the high order stabilization filter. Using this new method several reduced order filters are obtained. As a result, the cost of implementation is reduced and the reliability is increased. An algebraic method is also applied to redesign the stabilization filter so that the performance of the redesigned pitch control system is improved. In addition, the proposed dominant-data matching method can be applied to determine a reduced order model of a high order system. Unlike the reduced order models obtained by most existing model reduction methods, the reduced order model mentioned above has the exact assigned frequency-domain specifications of the original system. The dominant-data matching method can also be applied to identify any practical system.

ABSTRACT

A high-order stabilization filter was formerly designed to stabilize an unstable pitch control system of a terminal homing missile system. In this report, a new dominant-data matching method is presented to redesign the high order stabilization filter. Using this new method several reduced order filters are obtained. As a result, the cost of implementation is reduced and the reliability is increased. An algebraic method is also applied to redesign the stabilization filter so that the performance of the redesigned pitch control system is improved. In addition, the proposed dominant-data matching method can be applied to determine a reduced order model of a high order system. Unlike the reduced order models obtained by most existing model reduction methods, the reduced order model mentioned above has the exact assigned frequency-domain specifications of the original system. The dominant-data matching method can also be applied to identify any practical system.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

This report deals with the simplification and realization of a stabilization filter designed to stabilize the pitch control system of an unstable semi-active terminal homing missile system [1]. The block diagram of the existing stabilized system is shown in Fig. 1.

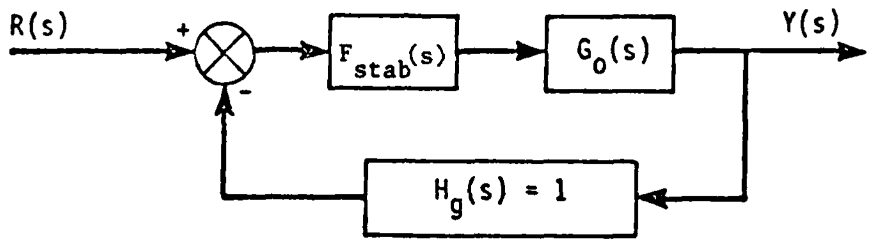

Figure 1. The Block Diagram of the Existing Control System

The overall transfer function of the existing system shown in Fig. 1 is given by

$$T_e(s) = \frac{F_{stab}(s)T_{act}(s)T_{miss}(s)}{1+F_{stab}(s)T_{act}(s)T_{miss}(s)H_g(s)}$$

$$\triangleq \frac{F_{stab}(s)G_0(s)}{1+F_{stab}(s)G_0(s)H_g(s)}$$

$$\triangleq \frac{G_e(s)}{1+G_e(s)H_g(s)} \tag{1.1}$$

where

$$F_{stab}(s) = \frac{1.6(\frac{s}{25}+1)(\frac{s}{125}+1)}{[(\frac{s}{150})^2+(\frac{0.6}{150})s+1][(\frac{s}{200})^2+(\frac{0.8}{200})s+1]}$$

$$= \frac{460800(s+25)(s+125)}{(s^2+90s+22500)(s^2+160s+4\times10^4)}$$

$$= \frac{460800(s+25)(s+125)}{(s+45\pm j143.0908802)(s+80\pm j183.3030278)} \qquad (1.2)$$

$G_0(s)$ = The transfer function of the actuator and the air frame dynamics of the missile system.

= The open loop transfer function of the original pitch control system iff $F_{stab}(s) = 1$ and $H_g(s) = 1$.

$= [T_{act}(s)]\ [T_{miss}(s)]$

$$= \left\{\frac{26937.9(s+65)(s+1500)}{(s+87.9\pm j95.5)(s+112.5)(s+1385)}\right\}\left\{\frac{12.04(s+0.1933)}{s(s-2.921)(s+3.175)}\right\}$$

$$= \frac{324332.316(s+0.1933)(s+65)(s+1500)}{s(s-2.921)(s+3.175)(s+87.9\pm j95.5)(s+112.5)(s+1385)} \qquad (1.3)$$

$$G_e(s) = F_{stab}(s)\ G_0(s)\ H_g(s) \qquad (1.4)$$

= The open loop transfer function of the existing stabilized system.

$H_g(s)$ = Transfer function of the gyro.

= 1, as the rate gyro is not present in the system.

After substituting $H_g(s) = 1$ and Eqn. (1.2) and (1.3) into Eqn. (1.1) it becomes

$$T_e(s) = \frac{G_e(s)}{1+G_e(s)} = \frac{b_0 s^{10}+b_1 s^9+\ldots+b_{10}}{a_0 s^{11}+a_1 s^{10}+a_2 s^9+\ldots+a_{11}} \triangleq \frac{N(s)}{D(s)} \tag{1.5}$$

where

$a_0 = 1$	$b_0 = 0$
$a_1 = 1.923554000\times10^3$	$b_1 = 0$
$a_2 = 9.316239040\times10^5$	$b_2 = 0$
$a_3 = 2.976950696\times10^8$	$b_3 = 0$
$a_4 = 6.231675318\times10^{10}$	$b_4 = 0$
$a_5 = 9.360329977\times10^{12}$	$b_5 = 1.494523312\times10^{11}$
$a_6 = 9.749923212\times10^{14}$	$b_6 = 2.563396371\times10^{14}$
$a_7 = 6.667397031\times10^{16}$	$b_7 = 5.017212044\times10^{16}$
$a_8 = 2.42040\ 431\times10^{18}$	$b_8 = 2.926344345\times10^{18}$
$a_9 = 2.911920560\times10^{18}$	$b_9 = 4.610004670\times10^{19}$
$a_{10} = 2.419047424\times10^{19}$	$b_{10} = 8.802158509\times10^{18}$
$a_{11} = 8.802158509\times10^{18}$	

From Fig. 1 as well as from Eqn. (1.2) it can be noticed that the existing stabilization filter $F_{stab}(s)$ is a fourth-order series compensator with two pairs of complex poles. $F_{stab}(s)$ is not a positive real function and hence cannot be synthesized with passive elements. The objective of this report is to develop computer-aided design methods for redesigning the stabilization filter in a simpler form so that the cost of implementation of the compensator can be reduced and at the same time the performance of the redesigned pitch control system remains the same as that of the existing pitch control system.

Nyquist plots of $G_e(s)$ and $G_0(s)$ are shown in Fig. 2. The dominant frequency-response data of $G_e(s)$ are given below:

i) The real and imaginary parts of $G_e(s)$ at $s = j\omega = j0$ are

$$Re\,[G_e(j0)] = -2.103817 \quad \text{and} \quad Im\,[G_e(j0)] = \infty \tag{2.1}$$

or $T_e(j0) = 1$

ii) The gain margin G_{em} of this system $G_e(j\omega)$ is

$$G_{em} = \left|\frac{1}{G_e(j\omega_{e\pi})}\right| = \left|\frac{1}{Re[G_e(j\omega_{e\pi})]}\right| \simeq \left|\frac{1}{-1.5}\right| \tag{2.2}$$

where the phase-crossover frequency $\omega_{e\pi}$ is given by

$$\omega_{e\pi} = 1.9 \text{ rad/sec such that } \angle G_e(j\omega_{e\pi}) = -180° \tag{2.3}$$

The equivalent real and imaginary parts of $G_e(j\omega_{e\pi})$ at $\omega_{e\pi} = 1.9$ rad/sec. are

$$Re[G_e(j\omega_{e\pi})] = -1.507944 \tag{2.4}$$

$$Im[G_e(j\omega_{e\pi})] = -0.006490205 \tag{2.5}$$

iii) The phase margin ϕ_{em} of the system $G_e(j\omega)$ is

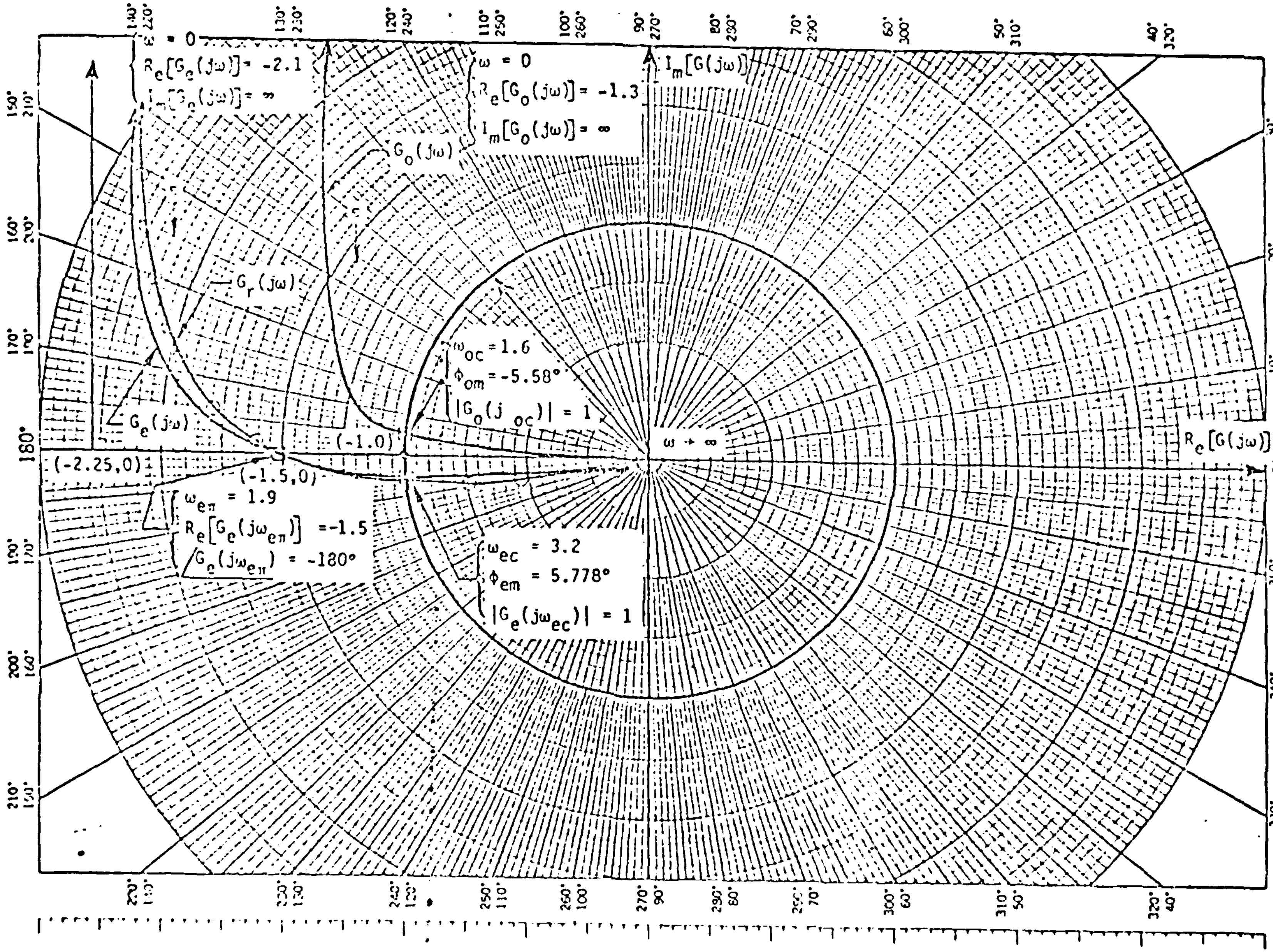

Figure 2. The Nyquist Plots of Various Open-Loop Transfer Functions

$$\phi_{em} = 180° + \underline{/G_e(j\omega_{ec})} \approx 5.7787° \tag{2.6}$$

where the gain cross-over frequency ω_{ec} is given by $\omega_{ec} \approx 3.2$ rad/sec so that

$$|G_e(j\omega_{ec})| = 1 \tag{2.7}$$

The equivalent real and imaginary parts of $G_e(j\omega)$ at $\omega = \omega_{ec} =$ 3.2 rad/sec. are

$$\mathrm{Re}[G_e(j\omega_{ec})] = -0.9939143 \tag{2.8}$$

$$\mathrm{Im}[G_e(j\omega_{ec})] = -0.09547478 \tag{2.9}$$

The frequency response data at $\omega = 0$ in (2.1) indirectly indicates the steady-state value of the unit step response of the system $T_e(s)$. The data at $\omega = \omega_{e\pi}$ and $\omega = \omega_{ec}$ in Eqn. (2) represent two control specifications [2]: gain margin and phase margin. These control specifications characterize the relative stability and the transient response of the existing stabilized system. The dominant frequency response data of $G_0(s)$, $F_{stab}(s)$ etc. are listed below:

i) The real and imaginary parts of $G_0(j\omega)$ at $\omega = 0$ are

$$\mathrm{Re}[G_0(j0)] = -1.304841 \quad \text{and} \quad \mathrm{Im}[G_0(j0)] = \infty \tag{3.1}$$

ii) The phase margin ϕ_{0m} of the original system $G_0(j\omega)$ is

$$\phi_{0m} = 180° + \underline{/G_0(j\omega_{0c})} = -5.58° \tag{3.2}$$

where the gain crossover frequency ω_{0c} is given by

$$\omega_{0c} \simeq 1.6 \text{ rad/sec. so that } |G_0(j\omega_{0c})| = 1. \tag{3.3}$$

Other frequency response data at $\omega_{e\pi} = 1.9$ rad/sec. and $\omega_{ec} = 3.2$ rad/sec. are

iii) $$\begin{aligned} Re[G_0(j\omega_{e\pi})] &= -0.9370766 \\ Im[G_0(j\omega_{e\pi})] &= \ \ 0.06716120 \end{aligned} \tag{3.4}$$

iv) $$\begin{aligned} Re[G_0(j\omega_{ec})] &= -0.6181657 \\ Im[G_0(j\omega_{ec})] &= \ \ 0.01949691 \end{aligned} \tag{3.5}$$

The dominant frequency response data of the stabilization filter $F_{stab}(s)$ are

i) $$Re[F_{stab}(j0)] = 1.6 \quad \text{and} \quad Im[F_{stab}(j0)] = 0 \tag{4.1}$$

ii) $$Re[F_{stab}(j\omega_{e\pi})] = 1.600492 \text{ and } Im[F_{stab}(j\omega_{e\pi})] = 0.1216316 \text{ at } \omega_{e\pi} = 1.9 \text{ rad/sec.} \tag{4.2}$$

or

$$|F_{stab}(j\omega_{e\pi})| = 1.605107127 \text{ and } \underline{/F_{stab}(j\omega_{e\pi})} = 4.345918198° \text{ at } \omega_{e\pi} = 1.9 \text{ rad/sec.} \tag{4.3}$$

iii) $$Re[F_{stab}(j\omega_{ec})] = 1.601402 \text{ and } Im[F_{stab}(j\omega_{ec})] = 0.2049554 \text{ at } \omega_{ec} = 3.2 \text{ rad/sec.} \tag{4.4}$$

or

$$|F_{stab}(j\omega_{ec})| = 1.614464333 \text{ and } \underline{/F_{stab}(j\omega_{ec})} = 7.293349493°$$
$$\text{at } \omega_{ec} = 3.2 \text{ rad/sec.} \tag{4.5}$$

Now, analyzing the data we have from Eqn. (1.3) and (1.4), it is clear that $G_0(s)$ and $G_e(s)$ are non-minimum phase functions and they are unstable because of the pole s = 2.821 which is in the right half plane of the s-plane. Referring to the Nyquist plots in Fig. 2, and according to Nyquist stability criterion the original missile system (without $F_{stab}(s)$) is unstable whereas the existing stabilized system is asymptotically stable. However, due to the small positive phase margin given in Eqn. (2.5), the time response of the existing stabilized system is quite oscillatory.

To redesign the pitch control system or the stabilization filter so that the cost of implementation is reduced and the flight control performance of the missile system is improved, two computer-aided methods have been developed. These two proposed methods will be discussed in this report step by step. In Chapter II a transfer function (called a standard transfer function $T_r(s)$) is obtained by using a dominant-data matching method. $T_r(s)$ matches the assigned specifications given in Eqn. (2). Therefore, the standard transfer function $T_r(s)$ mentioned above is a reduced-order model of the existing stabilized system $T_e(s)$ in Eqn. (1.5). The unit step response curves of $T_e(s)$ and $T_r(s)$ will be compared. This comparison will also verify that the data in Eqn. (2) are dominant ones.

To solve the nonlinear equations obtained in Chapter II four different methods of finding initial guesses are discussed in Chapter III.

In Chapter IV two reduced order models of the stabilization filter $F_{stab}(s)$ are obtained. One of these two is obtained by using the dominant-data matching method and the other by using a similar approach to fit a low order model that satisfies the specifications shown in Eqn. (4).

Chapter V consists of two parts, in the first part the dominant-data matching method is used to obtain an unstable reduced order model of the original high-order unstable system $G_0(s)$ shown in Eqn. (1.3). This is done just to simplify the design procedure. In the second part of Chapter V the algebraic method of Shieh [3] and Chen [4] is applied to redesign the pitch control system. This is done by designing a series filter in the feed forward path and a parallel filter in the feedback path. Thus, the advantages of feedback control structure have been fully utilized.

CHAPTER II

THE DOMINANT-DATA MATCHING METHOD

The design goals of a control system are often characterized by a set of control specifications. These specifications can be classified as i) time-domain specifications such as rise time, setting time, ii) frequency domain specifications such as phase margin, gain margin and iii) complex domain specifications such as damping ratio, and natural angular frequency. Empirical rules that link the specifications in the time, frequency, and complex domains have been developed by Truxal [5], Toro and Parker [6], Axelby [7] and Seshadri et. al. [8]. From these results, it is observed that most time-domain specifications and complex-domain specifications can be approximately converted to frequency-domain specifications. Some of these frequency-domain specifications are phase margin (ϕ_m), gain margin (G_m), maximum value of the closed-loop frequency response (M_p), phase crossover frequency (ω_π), gain-crossover frequency (ω_c), peak value frequency (ω_p), the bandwidth (ω_b), and the velocity error constant (K_v). Other important frequency response data are:

(1) The real part and imaginary part of the closed-loop function $T(s)$ as well as the corresponding open-loop function $G(s)$ at $s = j\omega = j0$,

(2) the real part of the open-loop transfer function $G(j\omega)$ at the phase crossover frequency ω_π which has been used to define the gain margin (G_m),

(3) the corner frequencies in the Bode plot of $G(j\omega)$ in the regions of $\omega = \omega_{c_1}$ where $20 \log|G(j\omega_{c1})| = +15$ db and $\omega = \omega_{c2}$ where $20 \log|G(j\omega_{c2})| = -15$ db.

Chen [9] has shown empirically that the open-loop poles and zeros of a system can be approximated by retaining the Bode plot in the regions of the ± 15 db boundaries.

The data at $\omega = 0$ often indicate the final value and the type of the system. More specifically, the value of $T(j0)$ or real part of $G(j0)$ indicates the final value of the system, while the imaginary part of $G(j0)$ indicates the type of the system. For example, for a type '0' system, the imaginary part of $G(j0)$ is 0, and for any system other than type '0', for example, say type '1', it is infinity.

Depending on the problem one has, one can use any one or a combination of the time domain, frequency domain and complex-domain specifications. However, in this case the original pitch control system that is available is a high order transfer function with large coefficients Eqn. (1.5). As a result the time response curve and the corresponding time domain specifications of this practical system $T_e(s)$ are difficult to obtain. On the other hand, with the help of a digital computer the frequency response curve and hence the frequency domain specifications of $T_e(s)$ can be easily determined. Therefore, a frequency domain approach or a dominant data matching method is proposed to construct a transfer function $T_r(s)$, a reduced order model of $T_e(s)$, and to redesign the pitch control system. Several methods have been already proposed [10, 11,12] to obtain reduced order models by considering frequency domain specifications. However, the only reduced order models that satisfy the assigned specifications exactly are the ones obtained by the proposed method.

From the rules of thumb it is observed that the gain margin, the phase margin, the gain cross-over frequency and the phase cross-over frequency are the most important frequency response data. These data are called the dominant frequency response data. Another important frequency response data is the steady state value of a closed-loop system, which is indirectly given by the value of the real part of the open loop transfer function $G(j\omega)$ at $\omega = 0$. These dominant frequency response data may be considered as the design goal. Let us assume that the desired reduced order model of $T_e(s)$ which may be called the standard model of $T_e(s)$ is

$$T_r(s) = \frac{b_0+b_1s+b_2s^2}{a_0+a_1s+a_2s^2+a_3s^3}$$

It is required that $T_r(s)$ satisfies all the conditions given in Eqn. (2).

From the conditions in (2.1), it may be observed that the system $T_e(s)$ is a type 1 system. Therefore $b_0 = a_0$. Also, to simplify the equation we let $a_3 = 1$. Thus, we have

$$T_r(s) = \frac{a_0+b_1s+b_2s^2}{a_0+a_1s+a_2s^2+s^3} = \frac{G_r(s)}{1+G_r(s)} \tag{5.1}$$

where the open-loop transfer function $G_r(s)$ is given by

$$G_r(s) = \frac{a_0+b_1s+b_2s^2}{s[(a_1-b_1)+(a_2-b_2)s+s^2]} = \frac{K[1+c_1s+c_2s^2]}{s[1+d_1s+d_2s^2]} \tag{5.2}$$

where $K = \dfrac{a_0}{a_1-b_1}$, $c_1 = \dfrac{b_1}{a_0}$, $c_2 = \dfrac{b_2}{a_0}$, $d_1 = \dfrac{a_2-b_2}{a_1-b_1}$, $d_2 = \dfrac{1}{a_1-b_1}$

The unknown coefficients a_i and b_i are to be determined by using the conditions in Eqn. (2). Following the basic definitions and substituting the assigned data in Eqn. (2) yields a set of nonlinear equations $f_i(a_0,a_1,a_2,b_1,b_2) = 0$ for $i = 1,2,\ldots,5$ as follows:

i) $$G_r(j\omega) = \frac{K[1+j\omega c_1+(j\omega)^2 c_2]}{j\omega[1+j\omega d_1+(j\omega)^2 d_2]}$$

$$= \frac{K}{j\omega}\,[1+j\omega(c_1-d_1)+(j\omega)^2(\qquad) + \ldots\ ]$$

$$\simeq \frac{K}{j\omega}\,[1+j\omega(c_1-d_1)]$$

$$= K(c_1-d_1) - j\,\frac{K}{\omega}$$

$$\lim_{\omega\to 0} G_r(j\omega) \simeq K(c_1-d_1) - j\infty$$

$$\mathrm{Re}[\lim_{\omega\to 0} G_r(j\omega)] \simeq K(c_1-d_1) = \frac{a_0}{a_1-b_1}\left(\frac{b_1}{a_0} - \frac{a_2-b_2}{a_1-b_1}\right) \tag{6.0}$$

Eqn. (2.1) gives $\mathrm{Re}[G_r(j0)] = -2.1$

or $$\frac{a_0}{a_1-b_1}\left(\frac{b_1}{a_0} - \frac{a_2-b_2}{a_1-b_1}\right) = -2.1$$

or $$\frac{b_1}{a_1-b_1} - \frac{a_0(a_2-b_2)}{(a_1-b_1)^2} = -2.1$$

or $$f_1(a_0,a_1,a_2,b_1,b_2) = b_1(a_1-b_1)-a_0(a_2-b_2)+2.1(a_1-b_1)^2 = 0 \tag{6.1}$$

ii) The data in (2.2), or $Re[G_r(j\omega_{e\pi})] = -1.5$, at $\omega_{e\pi} = 1.9$ rad/sec gives

$$Re[G_r(j\omega)]\Big|_{\omega=\omega_{e\pi}} = Re[\frac{(a_0-\omega^2 b_2)+j\omega b_1}{-\omega^2(a_2-b_2)+j\omega(a_1-b_1-\omega^2)}]\Big|_{\omega=\omega_{e\pi}=1.9} = -1.5$$

or

$$\frac{-\omega_{e\pi}^2(a_0-\omega_{e\pi}^2 b_2)(a_2-b_2)+\omega_{e\pi}^2 b_1(a_1-b_1-\omega_{e\pi}^2)}{\omega_{e\pi}^4(a_2-b_2)^2+\omega_{e\pi}^2(a_1-b_1-\omega_{e\pi}^2)^2}\Bigg|_{\omega_{e\pi}=1.9} = 1.5$$

or

$$f_2(a_0,a_1,a_2,b_1,b_2) = (a_2-b_2)(a_0-3.61b_2)-b_1(a_1-b_1-3.61) - 1.5[3.61(a_2-b_2)^2+(a_1-b_1-3.61)^2] = 0 \quad (6.2)$$

iii) The condition in (2.3), or $\underline{/G_r(j\omega_{e\pi})} = -180°$ where $\omega_{e\pi} = 1.9$ rad/sec, gives

$$\tan^{-1}\frac{\omega_{e\pi}b_1}{a_0-\omega_{e\pi}^2 b_2} - 180° + \tan^{-1}\frac{\omega_{e\pi}(a_1-b_1-\omega_{e\pi}^2)}{\omega_{e\pi}^2(a_2-b_2)} = -180°$$

or

$$\tan^{-1}\frac{\dfrac{1.9b_1}{a_0-3.61b_2}+\dfrac{a_1-b_1-3.61}{1.9(a_2-b_2)}}{1-\dfrac{1.9b_1(a_1-b_1-3.61)}{1.9(a_0-3.61b_2)(a_2-b_2)}} = 0°$$

or

$$f_3(a_0,a_1,a_2,b_1,b_2) = 3.61b_1(a_2-b_2)+(a_0-3.61b_2)(a_1-b_1-3.61) = 0 \quad (6.3)$$

iv) The data in (2.6) or

$$\phi_{em} = 180° + \underline{/G_r(j\omega_{ec})}\Big|_{\omega_{ec}=3.2 \text{ rad/sec}} = 5.7787°, \text{ yields}$$

$$180° + \tan^{-1}\frac{3.2b_1}{a_0-10.24b_2} - 180° + \tan^{-1}\frac{a_1-b_1-10.24}{3.2(a_2-b_2)} = 5.7787°$$

or

$$\tan^{-1}\frac{\dfrac{3.2b_1}{a_0-10.24b_2} + \dfrac{a_1-b_1-10.24}{3.2(a_2-b_2)}}{1 - \dfrac{3.2b_1(a_1-b_1-10.24)}{3.2(a_2-b_2)(a_0-10.24b_2)}} = 5.7787°$$

or

$$\frac{10.24b_1(a_2-b_2)+(a_0-10.24b_2)(a_1-b_1-10.24)}{3.2(a_2-b_2)(a_0-10.24b_2)-3.2b_1(a_1-b_1-10.24)} = 0.10120072$$

or

$$f_4(a_0,a_1,a_2,b_1,b_2) = 10.24b_1(a_2-b_2)+(a_0-10.24b_2)(a_1-b_1-10.24) - 0.3238423014[(a_2-b_2)(a_0-10.24b_2)-b_1(a_1-b_1 - 10.24)] = 0 \quad (6.4)$$

v) The condition in (2.7) or

$|G_r(j\omega_{ec})| = 1$ where $\omega_{ec} = 3.2$ rad/sec, gives

$$\left|\frac{a_0-10.24b_2+j3.2b_1}{-10.24(a_2-b_2)+j3.2(a_1-b_1-10.24)}\right| = 1$$

or

$$f_5(a_0,a_1,a_2,b_1,b_2) = (a_0-10.24b_2)^2+10.24b_1^2-104.8576(a_2-b_2)^2 - 10.24(a_1-b_1-10.24)^2 = 0 \quad (6.5)$$

Eqn. (6) is a set of high order simultaneous nonlinear algebraic equations which are very difficult to solve. The Newton-Raphson method that is available in most digital computers as a computer program package (called the z system [15])can be used to solve the nonlinear equations. However, it is well known that the Newton-Raphson method will only converge for a small range of starting values or the initial guesses. In order to improve the speed of convergence of the method four different methods of finding good initial guesses will be discussed in the next chapter.

CHAPTER III

THE INITIAL GUESS

In this report, the Newton-Raphson multidimensional method is suggested for solving nonlinear equations. However, as it is well known, high order nonlinear equations have many solutions and, depending on the starting values or the initial guesses, a solution may or may not be obtained. Therefore, the solution and the speed of convergence of a numerical method for solving nonlinear equations depend heavily on the initial guesses. In numerical mathematics, as well as in other areas of science, finding an appropriate initial guess for a set of nonlinear equations is itself a big problem to be solved. In this report, the following methods are proposed for good initial guesses. The applications of these methods depend on the type of problem one has.

(1) Initial guess by the synthesis method.

Suppose only the dominant frequency-response data in (2) are available and an approximate transfer function $T_r^*(s)$ of the desired $T_r(s)$ in (5.1) is required. The $T_r^*(s)$ is

$$T_r^*(s) = \frac{a_0^*+b_1^*s+b_2^*s^2}{a_0^*+a_1^*s+a_2^*s^2+s^3} \tag{7}$$

where a_i^* and b_i^* are the initial guesses of the numerical method. In the synthesis method $T_r^*(s)$ in (7) is obtained as follows:

<u>Step 1</u>. In this step a second-order approximate transfer function $T_2^*(s)$ is obtained by using $\phi_m = 5.7°$ and $\omega_c = 3.2$ rad/sec. in (2.6) and (2.7).

This $T_2^*(s)$ is

$$T_2^*(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{G_2^*(s)}{1+G_2^*(s)} \tag{8.1}$$

where ζ = the damping ratio and ω_n = the natural angular frequency. By following the basic definitions of ω_c and ϕ_m the following equations are obtained.

From (8.1)

$$G_2^*(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s}$$

$$G_2^*(j\omega) = \frac{\omega_n^2}{-\omega^2+j2\zeta\omega_n\omega} = \frac{\omega_n^2}{\sqrt{\omega^4+4\zeta^2\omega^2\omega_n^2}} \angle -180°+\tan^{-1}\frac{2\zeta\omega_n}{\omega}$$

By definition $|G_2^*(j\omega_c)| = 1$, where $\omega_c = 3.2$ rad/sec.

$$\therefore \quad \frac{\omega_n^2}{\sqrt{\omega_c^4+4\zeta^2\omega_c^2\omega_n^2}} = 1$$

or, $$\omega_n^4-40.96\zeta^2\omega_n^2-104.8576 = 0 \tag{8.2}$$

Next, by definition

$$\phi_m = \angle G_2^*(j\omega_c) + 180° = 5.7° \text{ given}$$

$$\therefore \quad 5.7° = -180° + \tan^{-1}\frac{2\zeta\omega_n}{\omega_c} + 180°$$

or, $$\frac{2\zeta\omega_n}{3.2} = \tan 5.7°$$

or $$\omega_n = \frac{0.1597012}{\zeta} \tag{8}$$

substituting (8.3) into (8.2) yields

$$\zeta^4 = 0.0000061422$$

the square root of which is

$$\zeta^2 = \pm 0.0024783561$$

considering the positive root only

$$\zeta = 0.0497830911 \quad \text{we neglect the negative root}$$

Substituting this in (8.3), yields

$$\omega_n = 3.207940617 \text{ rad/sec.}$$

$$\therefore \quad T_2^*(s) = \frac{10.290883}{s^2+0.3194024S+10.290883} \tag{8.4}$$

The poles that can be considered as dominant poles of a system can be determined from the characteristic equation in (8.1). As such dominant poles are

$$S_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -0.1598 \pm j3.2039$$

Step 2. In this step a third-order transfer function $T_3^*(s)$ is constructed by inserting a pole (s = -p) in it and modifying the term in the numerator of $T_2^*(s)$ so that the steady state value of the $T_3^*(s)$ is equal to unity, or

$$T_3^*(s) = \frac{p\,\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)(s+p)} = \frac{10.290883p}{(s^2+0.3194024S+10.290883)(s+p)}$$

$$= \frac{G_3^*(s)}{1+G_3^*(s)} \quad . \tag{8.5}$$

The unknown constant p is determined by using the condition in (2.2), or $\mathrm{Re}[G_3^*(j\omega_{e\pi})] = -1.5$, where $\omega_{e\pi} = 1.9$ rad/sec. Thus, from (8.5)

$$G_3^*(s) = \frac{10.290883p}{s^3+(p+0.3194024)s^2+(0.3194024p+10.290883)s} \tag{8.6}$$

let $s = j\omega$, then

$$G_3^*(j\omega) = \frac{10.290883p}{-\omega^2(p+0.3194024)+j\omega(0.3194024p+10.290883-\omega^2)} \tag{8.7}$$

$$\mathrm{Re}[G_3^*(j\omega)] = \frac{-10.290883p\,\omega^2(p+0.3194024)}{\omega^4(p+0.3194024)^2+\omega^2(0.3194024p+10.290883-\omega^2)^2} \tag{8.8}$$

at $\omega = \omega_{e\pi} = 1.9$ rad/sec, $\mathrm{Re}[G_3^*(j\omega)] = -1.5$

$$\therefore \quad \frac{-10.290883(p+0.3194024)}{3.61(p+0.3194024)^2+(0.3194024p+6.680883)} = -1.5 \tag{8.9}$$

After simplification, (8.9) becomes

$$p^2 - 1.391925823p - 14.29298735 = 0$$

or $\quad p = 4.540095027$, we neglect the negative root.

Thus (8.5) becomes

$$T_3^*(s) = \frac{46.72158673}{46.72158673+11.74100025S+4.859497427s^2+s^3} \tag{8.10}$$

Step 3. In this step another third-order transfer function $T_3^{**}(s)$ is constructed by inserting a zero in (8.10) as shown below.

$$T_3^{**}(s) = \frac{46.72158673+b_1^*s}{46.72158673+11.74100025s+4.859497427s^2+s^3} = \frac{G_3^{**}(s)}{1+G_3^{**}(s)} \tag{8.11}$$

The unknown constant b_1^* can be determined by using the condition in (6.0) and (2.1), or $Re[G_e(j0)] = -2.1$ as shown below. From (8.11), we get

$$G_3^{**}(s) = \frac{b_1^*s+46.72158673}{s^3+4.859497427s^2+(11.74100025-b_1)s}$$

or $$G_3^{**}(s) = \frac{46.72158673[1+\frac{b_1}{46.72158673}]}{s(11.74100025-b_1)[1+\frac{4.859497427}{11.74100025-b_1}s+\ldots]}$$

According to Eq. (6.0)

$$\lim_{\omega \to 0} Re[G_3^{**}(j\omega)] = \frac{46.72158673}{11.74100025-b_1}\left(\frac{b_1}{46.72158673} - \frac{4.859497427}{11.74100025-b_1}\right)$$

Given $\quad Re[G_3^{**}(j0)] = -2.1$ in (2.1)

$$\therefore \quad \frac{46.72158673}{11.74100025-b_1}\left(\frac{b_1}{46.72158673}-\frac{4.859497427}{11.74100025-b_1}\right) = -2.1$$

or $\qquad b_1^2-34.15563709b_1 + 56.76713817 = 0$

which gives $b_1 = 32.4037687$, since we are interested in the positive value only.

Substituting this into (8.11), we have

$$T_3^{**}(s) = \frac{46.7216+32.4038s}{46.7216+11.7410s+4.8595s^2+s^3} \qquad (8.12)$$

Equation (8.12) can be considered as an approximation of (7) by assuming $b_2^* = 0$. Thus the initial guesses in (7) are $a_0^* = 46.7216$, $a_1^* = 11.7410$, $a_2^* = 4.8595$, $b_1^* = 32.4038$, and $b_2^* = 0$. For solving Eq. (6.1)-(6.5) these constants are used as initial guesses for the Newton-Raphson method [15]. It is found that the numerical method converges at the 9th iteration with the error tolerance of 10^{-6}. The solutions of (6.1)-(6.5) are $a_0 = 6.378070$, $a_1 = 10.462220$, $a_2 = 1.259008$, $b_1 = 20.55667$ and $b_2 = 0.243466$. Therefore, the desired transfer function $T_r(s)$ is

$$T_r(s) = \frac{6.378070+20.55667s+0.243466s^2}{6.378070+10.462220s+1.259008s^2+s^3} \qquad (9)$$

The system represented by Eq. (9) has the exact frequency response data specified in (2).

(2) Initial guess by complex-curve fitting and continued fraction methods

In this part a simple method is presented to determine the approximate coefficients of a transfer function using the real parts and imaginary parts of the limited number of frequency-response data that are available. Using these data a low-order model is constructed. The low-order model is then expanded into a continued fraction of the second Cauer form to obtain a set of dominant quotients. Some non-dominant quotients are then inserted into the continued fraction to obtain an amplified-order model [16], which is the desired approximate transfer function $T_r^*(s)$ for the use of the initial guess.

Consider the transfer function

$$T_r^*(s) = \frac{b_0+b_1s+b_2s^2+\ldots+b_ms^m}{1+a_1s+a_2s^2+\ldots+a_ns^n} \tag{10.1}$$

where a_i and b_i are unknown coefficients to be determined. The problem of finding unknown coefficients of a transfer function as a ratio of two frequency-dependent polynomials has been investigated by Levy [17]. His method minimizes the sum of squares of the errors at arbitrary experimental points. However, for finding the unknown coefficients of a transfer function the method presented next is comparatively simple and straightforward.

Substituting $s = j\omega_k$ into (10.1), we have

$$T_r^*(j\omega_k) = \frac{b_0+j\omega_kb_1+(j\omega_k)^2b_2+\ldots+(j\omega_k)^mb_m}{1+j\omega_ka_1+(j\omega_k)^2a_2+\ldots+(j\omega_k)^na_n}$$

Separating the real parts and imaginary parts in the numerator and denominator of $T_r^*(j\omega_k)$ we have

$$T_r^*(j\omega_k) = \frac{(b_0-b_2\omega_k^2+b_4\omega_k^4-b_6\omega_k^6+\dots)+j(b_1\omega_k-b_3\omega_k^3+b_5\omega_k^5-b_1\omega_k^7+\dots)}{(1-a_2\omega_k^2+a_4\omega_k^4-a_6\omega_k^6+\dots)+j(a_1\omega_k-a_3\omega_k^3+a_5\omega_k^5-a_7\omega_k^7+\dots)}$$

$$= R(\omega_k) + jI(\omega_k)$$

$$= R_k + jI_k \tag{10.2}$$

where R_k and I_k are the given valuesof the real and imaginary parts of the $T_r^*(s)$ at the available frequencies ω_k. Multiplying both sides of (10.2) by the common denominator and separating the real and imaginary parts, we have

$$(b_0-b_2\omega_k^2+b_4\omega_k^4-b_6\omega_k^6+\dots)+j(b_1\omega_k-b_3\omega_k^3+b_5\omega_k^5-b_7\omega_k^7+\dots)$$

$$= R_k-a_2R_k\omega_k^2+a_4R_k\omega_k^4-a_6R_k\omega_k^6+\dots -a_1I_k\omega_k+a_3I_k\omega_k^3-a_5I_k\omega_k^5+\dots$$

$$+ j(a_1R_k\omega_k-a_3R_k\omega_k^3+a_5R_k\omega_k^5-a_7R_k\omega_k^7+\dots+I_k-a_2I_k\omega_k^2+a_4I_k\omega_k^4-a_6I_k\omega_k^6+\dots)$$

Equating the real and imaginary parts from both sides, yields

$$b_0-b_2\omega_k^2+b_4\omega_k^4-b_6\omega_k^6+\dots = R_k-a_2R_k\omega_k^2+a_4R_k\omega_k^4-a_6R_k\omega_k^6+\dots$$

$$- a_1I_k\omega_k+a_3I_k\omega_k^3-a_5I_k\omega_k^5+\dots \tag{10.3}$$

$$b_1\omega_k-b_3\omega_k^3+b_5\omega_k^5-b_7\omega_k^7+\dots = a_1R_k\omega_k-a_3R_k\omega_k^3+a_5R_k\omega_k^5-a_7R_k\omega_k^7+\dots$$

$$+ I_k-a_2I_k\omega_k^2+a_4I_k\omega_k^4-a_6I_k\omega_k^6+\dots \tag{10.4}$$

Eq. (10.3) and (10.4) can be written as

$$b_0-b_2\omega_k^2+b_4\omega_k^4-b_6\omega_k^6+\ldots+a_1I_k\omega_k+a_2R_k\omega_k^2-a_3I_k\omega_k^3-a_4R_k\omega_k^4+\ldots = R_k \tag{10.5}$$

$$b_1\omega_k-b_3\omega_k^3+b_5\omega_k^5-b_7\omega_k^7+\ldots-a_1R_k\omega_k+a_2I_k\omega_k^2+a_3R_k\omega_k^3-a_4I_k\omega_k^4-\ldots = I_k \tag{10.6}$$

In matrix form, (10.5) becomes

$$\begin{bmatrix} 1 & -\omega_1^2 & \omega_1^4 & -\omega_1^6 & \cdot & I_1\omega_1 & R_1\omega_1^2 & -I_1\omega_1^3 & -R_1\omega_1^4 & \cdot \\ 1 & -\omega_2^2 & \omega_2^4 & -\omega_2^6 & \cdot & I_2\omega_2 & R_2\omega_2^2 & -I_2\omega_2^3 & -R_2\omega_2^4 & \cdot \\ 1 & -\omega_3^2 & \omega_3^4 & -\omega_3^6 & \cdot & I_3\omega_3 & R_3\omega_3^2 & -I_3\omega_3^3 & -R_3\omega_3^4 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & -\omega_x^2 & \omega_x^4 & -\omega_x^6 & \cdot & I_x\omega_x & R_x\omega_x^2 & -I_x\omega_x^3 & -R_x\omega_x^4 & \cdot \end{bmatrix} \begin{bmatrix} b_0 \\ b_2 \\ b_4 \\ \cdot \\ a_1 \\ a_2 \\ \cdot \\ a_n \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ R_x \end{bmatrix} \tag{10.7}$$

where $x = n + \frac{m}{2} + 1$, if m is even

$= n + \frac{m+1}{2}$, if m is odd

Substituting a_i obtained in (10.7) into (10.6), we have another matrix equation to solve for b_i, $i = 1,3,5,\ldots$

$$\begin{bmatrix} \omega_1 & -\omega_1^3 & \omega_1^5 & -\omega_1^7 & \cdots \\ \omega_2 & -\omega_2^3 & \omega_2^5 & -\omega_2^7 & \cdots \\ \omega_3 & -\omega_3^3 & \omega_3^5 & -\omega_3^7 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \omega_y & -\omega_y^3 & \omega_y^5 & -\omega_y^7 & \cdots \end{bmatrix} \begin{bmatrix} b_1 \\ b_3 \\ b_5 \\ \cdot \\ b_k \end{bmatrix} = \begin{bmatrix} ((a_0I_1\omega_1^0+a_1R_1\omega_1^1)-(a_2I_1\omega_1^2+a_3R_1\omega_1^3)+\ldots) \\ ((a_0I_2\omega_2^0+a_1R_2\omega_2^1)-(a_2I_2\omega_2^2+a_3R_2\omega_2^3)+\ldots) \\ \cdot \\ \cdot \\ ((a_0I_y\omega_y^0+a_1R_y\omega_y^1)-(a_2I_y\omega_y^2+a_3R_y\omega_y^3)+\ldots) \end{bmatrix} \tag{10.8}$$

where $\omega_k^0 = 1$, $a_0 = 1$; $K = m$ and $y = \frac{m+1}{2}$ if m is odd; $K = m-1$ and $y = \frac{m}{2}$ if m is even.

In this pitch control system, the available data is given in (2) from which the following data is obtained,

$$\begin{aligned}
&\omega_1 = \omega_0 = 0\ , && R_1 = T_e(j0) = 1\ , && I_1 = 0 \\
&\omega_2 = \omega_{e\pi} = 1.9\ , && R_2 = Re[\frac{G_e(j\omega_{e\pi})}{1+G_e(j\omega_{e\pi})}] = 2.968398, && I_2 = I_m[\frac{G_e(j\omega_{e\pi})}{1+G_e(j\omega_{e\pi})}] = -0.02515098 \\
&\omega_3 = \omega_{ec} = 3.2\ , && R_3 = Re[\frac{G_e(j\omega_{ec})}{1+G_e(j\omega_{ec})}] = 0.3350731, && I_3 = I_m[\frac{G_e(j\omega_{ec})}{1+G_e(j\omega_{ec})}] = -10.43159
\end{aligned} \tag{11}$$

Data is available only at three frequencies, therefore the approximate transfer $T_2^*(s)$ is assumed to be

$$T_2^*(s) = \frac{b_0+b_1s}{1+a_1s+a_2s^2} \tag{12.1}$$

Substituting the data at ω_1, ω_2 and ω_3 in (11) into (10.7) yields

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -0.047786862 & 10.71591678 \\ 1 & -33.381088 & 3.431148544 \end{bmatrix} \begin{bmatrix} b_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2.968398 \\ 0.3350731 \end{bmatrix} \tag{12.2}$$

From (12.2), we get

$$b_0 = 1$$

$$b_0 - 0.047786862\,a_1 + 10.71591678a_2 = 2.968398 \tag{12.3}$$

$$b_0 - 33.381088a_1 + 3.431148544a_2 = 0.3350731 \tag{12.4}$$

Substituting $b_0 = 1$ into (12.3) and (12.4) yields

$$-0.047786862a_1 + 10.71591678a_2 = 1.968398$$

$$-33.381088a_1 + 3.431148544a_2 = -0.6649269$$

Solving these two equations, we get

$$a_1 = 0.0388179596$$

$$a_2 = 0.1838622891$$

Then substituting a_i and the data at ω_2 into (10.8) yields

$$3.2b_1 = 9.250106342$$

$$\therefore b_1 = 2.890658232$$

Substituting a_i and b_i, into (12.1) gives

$$T_2^*(s) = \frac{1+2.890658232s}{1+0.0388179596s+0.1838622891s^2} \qquad (12.5)$$

However, the desired approximate transfer function in (7) is a third-order function. Therefore $T_2^*(s)$ in (12.5) needs to be amplified. In this case this is done by using the continued fraction method [16] as follows.

$T_2^*(s)$ is first expanded into a continued fraction of the second Cauer form to obtain a set of dominant quotients. They are given as

$h_1 = 1$, $h_2 = -0.3506507744$, $h_3 = 0.9650474175$ and $h_4 = 16.072551656$. Then the order of $T_2^*(s)$ is amplified by inserting two nondominant quotients $h_5 = 100$ and $h_6 = 0.1$, or

$$T_2^*(s) = \frac{1+2.890658232S}{1+0.0388179596S+0.1838622891s^2}$$

$$= \cfrac{1}{h_1+\cfrac{1}{\cfrac{h_2}{s}\cfrac{1}{h_3+\cfrac{1}{\cfrac{h_4}{s}}}}}$$

$$\simeq \cfrac{1}{h_1+\cfrac{1}{\cfrac{h_2}{s}\cfrac{1}{h_3+\cfrac{1}{\cfrac{h_4}{s}+\cfrac{1}{h_5+\cfrac{1}{\cfrac{h_6}{s}}}}}}} \qquad (12.6)$$

Substituting

$h_1 = 1$

$h_2 = -0.3506507744$

$h_3 = -0.9650474175$

$h_4 = 16.07251656$

$h_5 = 100$

$h_6 = 0.1$ into (12.6), it becomes

$$T_2^*(s) \simeq T_3^*(s) = T_r^*(s) \simeq \frac{54.3885+162.6914s+15.8219s^2}{54.3885+7.5839s+10.2146s^2+s^3} \tag{12.7}$$

In solving (6.1)-(6.5) if we use the coefficients in (12.7) as initial guesses; $a_0^* = 54.3885$, $a_1^* = 7.5839$, $a_2^* = 10.2146$, $b_1^* = 162.6914$ and $b_2^* = 15.8219$, we have the desired coefficients in (9) at the 15th iteration [15] with the error tolerance of 10^{-6}. This proves once again that if the inserted positive quotients $h_i >> 1$ and $h_{i+1} << 1$ (i is an odd number) the amplified order model is a good approximation of the original low-order model.

(3) Initial guess by continued fraction method [18]

Shieh [3] and Chen [10] have proposed a continued fraction method for model reductions. In this case their method is utilized to find initial guesses to solve Eq. (6.1)-(6.5). The numerator polynomial N(s) and the denominator polynomial D(s) in Eq. (1.5) are arranged in ascending order and expanded into the continued fraction of the second Cauer form by performing repeated long divisions as follows.

$$T_e(s) = \frac{N(s)}{D(s)} = \frac{b_{10}+b_9s+b_8s^2+\ldots+b_0s^{10}}{a_{11}+a_{10}s+a_9s^2+\ldots+a_0s^{11}} \quad \text{where } a_i,\ b_i \text{ are given in (1.5)}$$

$$= \cfrac{1}{h_1+\cfrac{s}{h_2+\cfrac{s}{h_3+\cfrac{s}{h_4+\cfrac{s}{h_5+\cfrac{s}{h_6+\ddots}}}}}} \tag{13.0}$$

where $h_1 = 1$

$$h_2 = -0.401749$$
$$h_3 = -0.475321$$
$$h_4 = 25.1998$$
$$h_5 = -0.0322195 \qquad (13.1)$$
$$h_6 = -24.1061$$
$$h_7 = \ldots$$
$$\vdots$$
$$h_{22} = \ldots$$

The reduced order models of $T_e(s)$ can be obtained by retaining the first few dominant quotients, $h_i = 1,2,\ldots$ The number of quotients used depends on the order and form of the reduced model. This is explained below

$$T_e(s) \simeq \cfrac{1}{h_1 + \cfrac{s}{h_2}} = \frac{h_2}{h_1 h_2 + s} \qquad (13.2)$$

$$\simeq \cfrac{1}{h_1 + \cfrac{s}{h_2 + \cfrac{s}{h_3}}} = \frac{h_2 h_3 + s}{h_1 h_2 h_3 + (h_1 + h_3)s} \qquad (13.3)$$

$$\simeq \cfrac{1}{h_1 + \cfrac{s}{h_2 + \cfrac{s}{h_3 + \cfrac{s}{h_4}}}} = \frac{h_2 h_3 h_4 + (h_2 + h_4)s}{h_1 h_2 h_3 h_4 + (h_1 h_2 + h_1 h_4 + h_3 h_4)s + s^2} \qquad (13.4)$$

$$\simeq \cfrac{1}{h_1+\cfrac{s}{h_2+\cfrac{s}{h_3+\cfrac{s}{h_4+\cfrac{s}{h_5}}}}} = \frac{h_2h_3h_4h_5+(h_2h_3+h_2h_5+h_4h_5)s+s^2}{h_1h_2h_3h_4h_5+(h_1h_2h_3+h_1h_2h_5+h_1h_4h_5+h_3h_4h_5)s+(h_1+h_3+h_5)s^2} \tag{13.5}$$

$$\simeq \cfrac{1}{h_1+\cfrac{s}{h_2+\cfrac{s}{h_3+\cfrac{s}{h_4+\cfrac{s}{h_5+\cfrac{s}{h_6}}}}}} = \frac{h_2h_3h_4h_5h_6+(h_2h_3h_4+h_2h_3h_6+h_2h_5h_6+h_4h_5h_6)s+(h_2+h_4+h_5)s^2}{h_1h_2h_3h_4h_5h_6+(h_1h_2h_3h_4+h_1h_2h_3h_6+h_1h_2h_5h_6+h_1h_4h_5h_6+h_3h_4h_5h_6)s+(h_1h_2+h_1h_4+h_1h_6+h_3h_4+h_3h_6+h_5h_6)s^2+s^3} \tag{13.6}$$

$$\simeq \ldots$$

Substituting the h_i's in (13.1) into (13.6) yields the third-order approximate model of $T_e(s)$ as follows:

$$T^*_{3c}(s) = \frac{3.7376+10.4692s+0.6920s^2}{3.7376+10.1661s+0.9488s^2+s^3} \tag{14}$$

Using the coefficients in (14.1) as the initial guesses: $a^*_0 = 3.7376$, $a^*_1 = 10.1661$, $a^*_2 = 0.9488$, $b^*_1 = 19.4692$ and $b^*_2 = 0.6920$, the desired solution($T_r(s)$ given in (9))of the set of nonlinear equation (6.1)-(6.5) are obtained at the 8th iteration with the error tolerance of 10^{-6}.

As it has just been shown in this particular case, the continued fraction method of finding the initial guess has worked out nicely. However, this is not true always. For example, if the reduced order model by the continued fraction method turns out to be an unstable system, the coefficients of such a reduced order model cannot be used to solve a set of nonlinear equations. Because, an unstable initial guess often leads to solutions which will give rise to an unstable system only. In such cases the following mixed method can be used to obtain a stable reduced-order model for approximation.

(4) Initial guess by using the mixed method.

In this section of the report two mixed methods are discussed. One has the advantages of both the continued fraction method [3,10] and the dominant pole method [19]. The other has the advantages of the continued fraction as well as the Routh table [11], from which the equivalent dominant-poles can be obtained. The reduced-order models obtained by the mixed method are stable and can be used as good initial guesses.

The relationship between the quotients h_i and the coefficients a_i and b_i in (13.0) can be expressed by the following matrix Eq. [3,4]:

$$[b] = [H]\,[a] \tag{15}$$

where
$$[a]^T = [a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0] \ ,$$
$$[b]^T = [b_{n-1}, b_{n-2}, \ldots, b_2, b_1, b_0] \ ,$$
$$[H] = [H_2]^{-1}\,[H_1] \ ,$$

here T designates transpose of a matrix

$$[H_2] = \begin{bmatrix} h_1 & 0 & 0 & . & 0 & 0 \\ 1 & h_2 & 0 & . & 0 & 0 \\ 0 & 1 & h_3 & . & 0 & 0 \\ 0 & 0 & 1 & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 1 & h_n \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & . & 0 & 0 \\ 0 & h_1 & 0 & . & 0 & 0 \\ 0 & 1 & h_2 & . & 0 & 0 \\ 0 & 0 & 1 & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 1 & h_{n-1} \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 & 0 & . & 0 & 0 \\ 0 & 1 & 0 & . & 0 & 0 \\ 0 & 0 & 1 & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 0 & h_1 \end{bmatrix}$$

$$[H_1] = \begin{bmatrix} 1 & 0 & 0 & . & 0 & 0 \\ 0 & h_2 & 0 & . & 0 & 0 \\ 0 & 1 & h_3 & . & 0 & 0 \\ 0 & 0 & 1 & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 1 & h_n \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & . & 0 & 0 \\ 0 & 1 & 0 & . & 0 & 0 \\ 0 & 0 & h_2 & . & 0 & 0 \\ 0 & 0 & 1 & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 1 & h_{n-1} \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 & 0 & . & 0 & 0 \\ 0 & 1 & 0 & . & 0 & 0 \\ 0 & 0 & 1 & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 0 & h_2 \end{bmatrix}$$

Consider the reduced-order model of the original system as

$$T_r(s) = \frac{e_0+e_1s+\ldots+e_{r-1}s^{r-1}}{d_0+d_1s+\ldots+d_{r-1}s^{r-1}+d_rs^r} \quad , \quad d_r = 1 \tag{16}$$

The denominator polynomial in (10) is approximated by the product of the dominant poles of the original system $T_e(s)$. Thus d_i is known. Replacing a_i and b_i in (15) by d_i and e_i in (16), respectively, Eq. (15) can be solved for e_i in (16). The $T_r(s)$ obtained has the dominant poles and the dominant quotients of $T_e(s)$ and it is always stable, therefore, $T_r(s)$ can be used as a good initial guess in solving (6.1)-(6.5).

In case the roots of D(s) in (13.0) are not available, the approximate equivalent dominant poles and the resulting coefficients d_i can be determined from the Routh table as suggested by Hutton and Friedland[9]. The steps involved are explained below.

$$\text{Assume } T(s) = \frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\ldots+b_2+b_1+b_0}{a_ns^n+a_{n-1}s^{n-1}+\ldots+a_2+a_1+a_0} = \frac{n(s)}{d(s)} \qquad (17.1)$$

is the original transfer function for which the reduced order model is needed.

Step 1. Construct a Routh array [20] using the coefficients a_i of the d(s) above and the Routh algorithm. The Routh array is shown below. To obtain a general algorithm a_i is expressed double-subscripted notation, for example, $a_{i,j}$

$$\begin{array}{lllll}
 & a_{11} \triangleq a_n & a_{12} \triangleq a_{n-2} & a_{13} \triangleq a_{n-4} & \cdots \quad a_0 \\
\gamma_1 = \frac{a_{11}}{a_{21}} & & & & \\
 & a_{21} \triangleq a_{n-1} & a_{22} \triangleq a_{n-3} & a_{23} \triangleq a_{n-5} & \cdots \\
\gamma_2 = \frac{a_{21}}{a_{31}} & & & & \\
 & a_{31} \triangleq a_{12}-\gamma_1 a_{22} & a_{32} \triangleq a_{13}-\gamma_1 a_{22} & a_{33} & \cdots \\
\gamma_3 = \frac{a_{31}}{a_{41}} & & & & \\
 & a_{41} \triangleq a_{22}-\gamma_2 a_{32} & a_{42} \triangleq a_{23}-\gamma_2 a_{33} & \cdot & \cdots \\
 & & & & \\
 & \cdot \quad \cdot \quad \cdot & & &
\end{array}$$

$$\begin{array}{lll}
 & a_{n-2,1} & a_{n-2,2} \\
\gamma_{n-2} = \dfrac{a_{n-2,1}}{a_{n-1,1}} & & \\
 & a_{n-1,1} & a_{n-1,2} = a_0 \\
\gamma_{n-1} = \dfrac{a_{n-1,1}}{a_{n,1}} & & \\
 & a_{n,1} & \\
\gamma_{n} = \dfrac{a_{n,1}}{a_{n+1,1}} & & \\
 & a_{n+1,1} = a_0 &
\end{array} \qquad (17.2)$$

In general

$$\begin{aligned} a_{i,j} &= a_{i-2,j+1} - \gamma_{i-2} a_{i-1,j+1}; \; i = 1,2,\ldots, j = 3,4,\ldots \\ \gamma_i &= a_{i,1}/a_{i+1,1} \end{aligned} \qquad (17.3)$$

Step 2. In this step various approximate low-order polynomials $d_i^*(s)$ are constructed from any two consecutive rows in the Routh array, for example, say from the last row and the next to the last now and so on. This is explained below.

The first order (i = 1) approximate equation is

$$d_1^*(s) = a_{n,1}s + a_{n+1,1} = a_{n,1}s + a_0 = 0 \qquad (17.4)$$

The second order (i = 2) approximate equation is

$$d_2^*(s) = a_{n-1,1}s^2 + a_{n,1}s + a_{n-1,2} = a_{n-1,1}s^2 + a_{n,1}s + a_0 = 0 \qquad (17.5)$$

The third order (i = 3) approximate equation is

$$d_3^*(s) = a_{n-2,1}s^3+a_{n-1,1}s^2+a_{n-2,2}s+a_{n-1,2} = 0$$

$$= a_{n-2,1}s^3+a_{n-1,1}s^2+a_{n-2,2}s+a_0 = 0 \qquad (17.6)$$

and so on.

When the original system (17.1) is asymptotically stable, all γ_i are positive values and the approximate polynomials $d_i^*(s)$ are the Hurwitz polynomials. The $d_i^*(s)$ are normalized simply by dividing each coefficient in $d_i^*(s)$ by the coefficient of the highest order term in s. These normalized $d_i^*(s)$ are the denominator polynomials of the reduced-order models $T_i^*(s)$ of the original system. Then the numerator polynomials of $T_i^*(s)$ are determined simply by substituting the coefficients of $d_{i_{norm}}^*(s)$ in place of [a] in (15) and then solving the matrix equation (15) for [b], which are the coefficients of the numerator polynomial of $T_i^*(s)$.

The third-order reduced order model $T_{3m}^*(s) = \dfrac{n_3^*(s)}{d_3^*(s)}$ of the original pitch control system in (1.5) obtained by using the mixed method is explained below.

At first, the Routh array of the pitch control system in (1.5) is obtained. From the Routh array the normalized approximate denominator $d_3^*(s)$ is found.

$$d_3^*(s) = s^3+0.9523822967s^2+10.19241445s+3.745517989 \qquad (17.7)$$

To determine $n_3^*(s) \triangleq b_2s^3+b_1s+b_0$, the coefficients of $d_3^*(s)$ are substituted into (15) as shown below.

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} H \end{bmatrix} \begin{bmatrix} 3.7455 \\ 10.1924 \\ 0.9524 \end{bmatrix} \tag{17.8}$$

or
$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} H_2 \end{bmatrix}^{-1} \begin{bmatrix} H_1 \end{bmatrix} \begin{bmatrix} 3.7455 \\ 10.1924 \\ 0.9524 \end{bmatrix}$$

or
$$\begin{bmatrix} H_2 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} H_1 \end{bmatrix} \begin{bmatrix} 3.7455 \\ 10.1924 \\ 0.9524 \end{bmatrix}$$

or
$$\begin{bmatrix} h_1 & 0 & 0 \\ 1 & h_2 & 0 \\ 0 & 1 & h_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & h_1 & 0 \\ 0 & 1 & h_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & h_1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 1 & h_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & h_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3.7455 \\ 10.1924 \\ 0.9524 \end{bmatrix}$$

or
$$\begin{bmatrix} h_1 & 0 & 0 \\ 1 & h_1h_2 & 0 \\ 0 & h_1+h_3 & h_1h_2h_3 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 1 & h_2h_3 \end{bmatrix} \begin{bmatrix} 3.7455 \\ 10.1924 \\ 0.9524 \end{bmatrix} \tag{17.9}$$

where the h_i's are the quotients of $T_e(s)$ in (1.5), which are given in (13.1). Substituting the values of h_1, h_2 and h_3 from (13.1) into (17.9) and then simplifying, we get

$$b_0 = 3.7455 \tag{18.1}$$

$$b_0 - 0.401749\ b_1 = -4.094787$$

substituting (18.1) yields

$$b_1 = 19.5154 \tag{18.2}$$

and $\quad 0.524679\ b_1 + 0.19096\ b_2 = 10.37427$

Substituting (18.2) yields

$$b_2 = 0.7066 \tag{18.3}$$

Therefore

$$T^*_{3m}(s) = \frac{0.7066s^2+19.5154s+3.7455}{s^3+0.9524s^2+10.1924s+3.7455} \tag{19}$$

In solving the nonlinear Eqs. (6.1)-(6.5) if the coefficient of $T^*_{3m}(s)$ in (19) are used as starting values: $a^*_0 = 3.7455$, $a^*_1 = 10.1924$, $a^*_2 = 0.9524$, $b^*_1 = 19.5154$ and $b^*_2 = 0.7066$; the Newton-Raphson method [15] converges to the desired solution in (9) or

$$T_r(s) = \frac{6.37807+20.55661s+0.243466s^2}{6.37807+10.46222s+1.259008s^2+s^3} = \frac{G_r(s)}{1+G_r(s)}$$

at the 8th iteration with the error tolerance of 10^{-6}.

From (9)

$G_r(s)$ = The open-loop transfer function of the standard model $T_r(s)$.

$$= \frac{6.37807+20.55661s+0.24346s^2}{s(-10.09439+1.015542s+s^2)} \qquad (20)$$

The Nyquist plot of $G_r(s)$ is shown in Fig. 2 and the unit step responses of $T_r(s)$, $T_{3c}^*(s)$, $T_{3m}^*(s)$ and $T_e(s)$ are compared in Fig. 3. All three reduced-order models $T_r(s)$, $T_{3c}^*(s)$ and $T_{3m}^*(s)$ give very satisfactory approximate time response curves. However, only the $T_r(s)$ in (9), which uses the method of dominant frequency response data matching, has the exact dominant-frequency response data as the original system $T_e(s)$ given in (2).

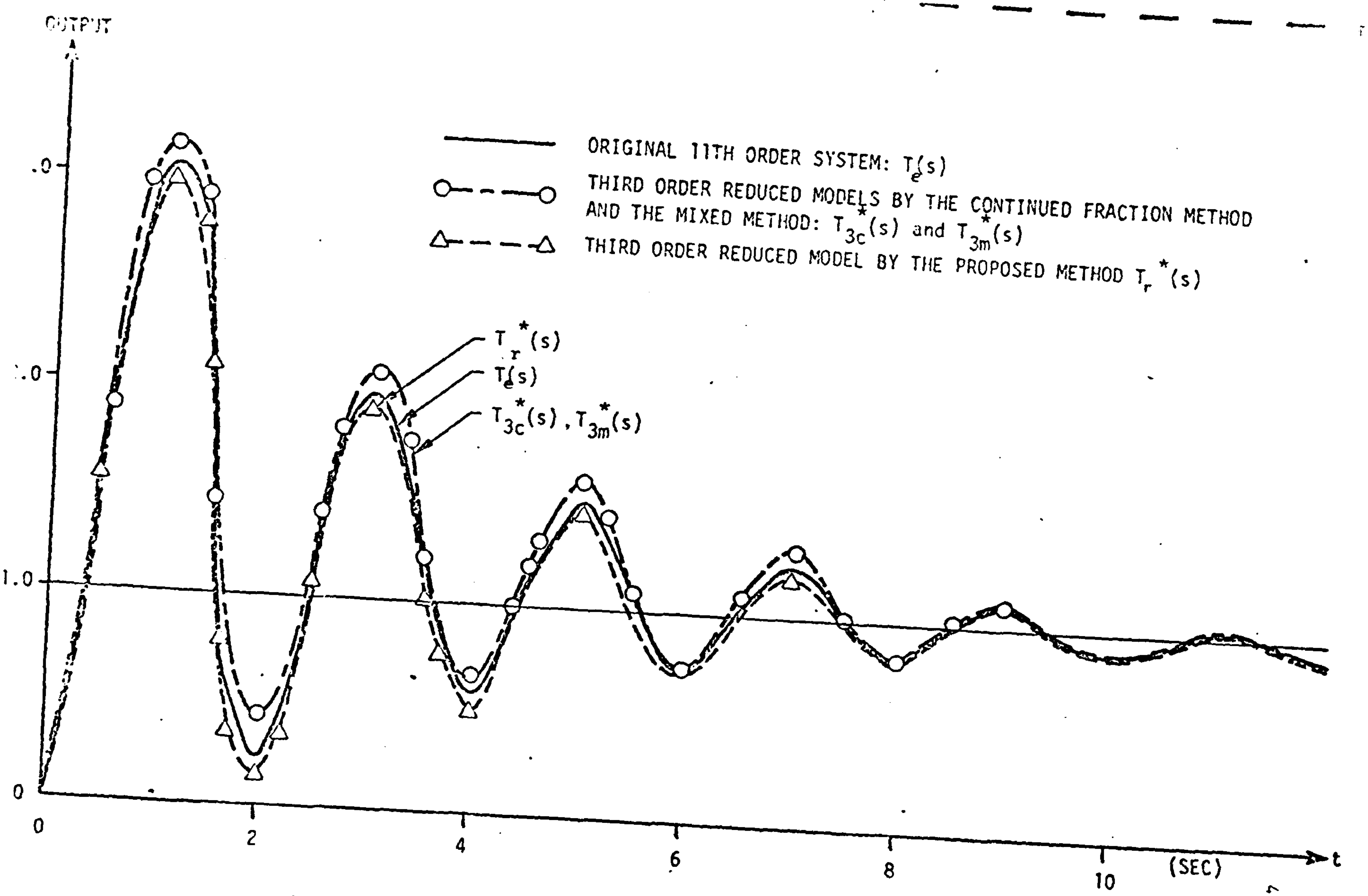

FIGURE 3. TIME RESPONSES OF ORIGINAL AND THIRD ORDER REDUCED MODELS.

CHAPTER IV

SIMPLIFICATION OF THE EXISTING STABILIZATION FILTER

As it appears from its name, the purpose of the stabilization filter is to stabilize the original unstable system. The transfer function of the existing stabilization filter $F_{stab}(s)$ is known and is given in (1.2). As it is mentioned in the introduction of this report, the objective of this report is to redesign the stabilization filter so that the cost of implementation can be reduced and at the same time the performance of the redesigned pitch control system is the same as that of the existing stabilized pitch control system.

In this chapter two different transfer functions are obtained for the stabilization filter. Both of these transfer functions are obtained by direct simplification of the available transfer function of $F_{stab}(s)$, and one of them is obtained by using the dominant data matching method of Chapter II.

The $F_{stab}(s)$ in (1.2) can be considered as the closed-loop transfer function of a control system as

$$F_{stab}(s) = \frac{N_s(s)}{D_s(s)} = \frac{G_{stab}(s)}{1+G_{stab}(s)} = \frac{460800s^2+69120000s+144\times10^7}{s^4+250s^3+76900s^2+72\times10^5s+9\times10^8} \quad (21.1)$$

where the open-loop transfer function $G_{stab}(s)$ is

$$G_{stab}(s) = \frac{460800s^2+69120000s+144\times10^7}{s^4+250s^3-383900s^2-61920000s-5.4\times10^8} \quad (21.2)$$

The dominant frequency-response data of this system are given below.

i) $$G_{stab}(j0) = -\frac{1}{0.375} \tag{22.1}$$

ii) $$Re[G_{stab}(j\omega_{s\pi})] = -1.032833 \tag{22.2}$$

$$Im[G_{stab}(j\omega_{s\pi})] = 0.002017351 \tag{22.3}$$

where $\omega_{s\pi}$ = The phase crossover frequency of the stabilization filter

= 140 rad/sec.

iii) $$Re[G_{stab}(j\omega_{sc})] = -1.002941 \tag{22.4}$$

$$Im[G_{stab}(j\omega_{sc})] = -0.03668759 \tag{22.5}$$

where ω_{sc} = The gain crossover frequency of the stabilization filter

= 200 rad/sec.

Suppose the reduced-order model $F_{s1}(s)$ of the stabilization filter is

$$F_{s1}(s) = \frac{b_0+b_1s}{a_0+a_1s+s^2} = \frac{G_{s1}(s)}{1+G_{s1}(s)} \tag{23.1}$$

where $G_{s1}(s)$ = The open-loop transfer function of $F_{s1}(s)$

$$= \frac{b_0+b_1s}{(a_0-b_0)+(a_1-b_1)s+s^2} \tag{23.2}$$

The constants a_i and b_i are unknown constants to be determined. Using the specifications given in (22) and following the basic definitions of those specifications the unknown constants a_i and b_i are determined as shown below.

For $F_{s1}(s)$ in (23.1) to be a reduced order model of $F_{stab}(s)$, $G_{s1}(s)$ must satisfy all the specifications of $G_{stab}(s)$ in (22). Applying the condition in (22.1) to the system $G_{s1}(s)$ in (23.2) yields

$$\text{at } s = j0 \quad G_{s1}(j0) = \frac{b_0}{a_0-b_0} = -\frac{1}{0.375}$$

$$\text{or,} \quad b_0 = 1.6a_0 \tag{24.1}$$

Substituting (24.1) into (23.1) and (23.2), respectively, we get

$$F_{s1}(s) = \frac{1.6a_0+b_1s}{a_0+a_1s+s^2} \tag{24.2}$$

and

$$G_{s1}(s) = \frac{1.6a_0+b_1s}{-0.6a_0+(a_1-b_1)s+s^2} \tag{24.3}$$

$$\text{at } s = j\omega \quad G_{s1}(j\omega) = \frac{1.6a_0+j\omega b_1}{-(0.6a_0+\omega^2)+j\omega(a_1-b_1)}$$

$$= \frac{(1.6a_0+j\omega b_1)[-(0.6a_0+\omega^2)-j\omega(a_1-b_1)]}{(0.6a_0+\omega^2)^2+\omega^2(a_1-b_1)^2}$$

$$\therefore \quad Re[G_{s1}(j\omega)] = \frac{-1.6a_0(0.6a_0+\omega^2)+\omega^2b_1(a_1-b_1)}{(0.6a_0+\omega^2)^2+\omega^2(a_1-b_1)^2} \tag{24.4}$$

$$Im[G_{s1}(j\omega)] = \frac{-\omega b_1(0.6a_0+\omega^2)-1.6\omega a_0(a_1-b_1)}{(0.6a_0+\omega^2)^2+\omega^2(a_1-b_1)^2} \tag{24.5}$$

i) Specification in (22.2) yields

$$\mathrm{Re}[G_{s1}(j140)] = -1.032833$$

Substituting (24.4) above gives

$$\frac{-1.6a_0(0.6a_0+19600)+19600b_1(a_1-b_1)}{(0.6a_0+19600)^2+19600(a_1-b_1)^2} = -1.032833$$

or $$f_1(a_0,a_1,b_1) = -1.6a_0(0.6a_0+19600)+19600b_1(a_1-b_1)$$
$$+ 1.032833[(0.6a_0+19600)^2+19600(a_1-b_1)^2] = 0 \tag{25.1}$$

ii) The data in (22.3) when applied to (24.5) yields

$$\mathrm{Im}[G_{s1}(j140)] = 0.002017351$$

or, $$\frac{-140b_1(0.6a_0+19600)-224a_0(a_1-b_1)}{(0.6a_0+19600)^2+19600(a_1-b_1)^2} = 0.002017351$$

or $$f_2(a_0,a_1,b_1) = -140b_1(0.6a_0+19600)-224a_0(a_1-b_1)$$
$$- 0.002017351[(0.6a_0+19600)^2+19600(a_1-b_1)^2] = 0 \tag{25.2}$$

iii) The data in (22.4) when applied to (24.4) gives

$$\mathrm{Re}[G_{s1}(j200)] = -1.002941$$

or $$\frac{-1.6a_0(0.6a_0+40000)+40000b_1(a_1-b_1)}{(0.6a_0+40000)^2+40000(a_1-b_1)^2} = -1.002941$$

or $$f_3(a_0,a_1,b_1) = -1.6a_0(0.6a_0+40000)+40000b_1(a_1-b_1)$$

$$+\ 1.002941[(0.6a_0+40000)^2+40000(a_1-b_1)^2] = 0 \tag{25.3}$$

Equation (25) is a set of nonlinear equations. The unknown constants a_i and b_i in (23.1) are determined by solving (25). However, to solve equation (25) the proper initial guesses have to be determined first. As discussed in Chapter III, the initial guesses can be determined from the reduced-order model of the existing stabilization filter $F_{stab}(s)$ in (1.2). Using the mixed method of the continued fraction approximation and the Routh approximation, a reduced-order model $F^*_{r1}(s)$ is obtained as follows.

Assume $$F^*_{r1}(s) = \frac{b^*_0+b^*_1 s}{a^*_0+a^*_1 s+s^2} = \frac{n^*(s)}{d^*(s)} \tag{26.1}$$

Routh's st ility criterion for $F_{stab}(s)$ is

s^4	1	76900	9×10^8
s^3	250	72×10^5	
s^2	48100	9×10^8	
s^1	2522245.322		
s^0	9×10^8		

As discussed in the Section 4 of Chapter III, the $d^*(s)$ in (26.1) is approximated from the Routh criterion shown above. Thus

$$d^*(s) = 48100s^2+2522245.322s+9\times10^8 = 0$$

After normalization $d^*(s)$ becomes

$$d^*(s) = s^2+52.4375s+18711.01871 \tag{26.2}$$

Therefore, now (26.1) becomes

$$F^*_{r1}(s) = \frac{b_1^*s+b_0^*}{s^2+52.4375s+18711.01871} \tag{26.3}$$

The quotients h_i of $F_{stab}(s)$ are obtained below

$$\begin{array}{lccccc}
 & 9\times10^8 & 72\times10^5 & 76900 & 250 & 1 \\
h_1 = 0.625 & & & & & \\
 & 144\times10^7 & 69120000 & 460800 & & \\
h_2 = -40 & & & & & \\
 & -36\times10^6 & -211100 & 250 & 1 & \\
h_3 = -0.593315 & & & & & \\
 & 60676000 & 470800 & 40 & & \\
\vdots & \vdots & & & &
\end{array}$$

Using Eq. (15) b_0^* and b_1^* in (26.1) are determined as follows.

$$\begin{bmatrix} h_1 & 0 \\ 1 & h_1h_2 \end{bmatrix} \begin{bmatrix} b_0^* \\ b_1^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & h_2 \end{bmatrix} \begin{bmatrix} 18711.01871 \\ 52.84375 \end{bmatrix}$$

or

$$\begin{bmatrix} 0.625 & 0 \\ 1 & -25 \end{bmatrix} \begin{bmatrix} b_0^* \\ b_1^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -40 \end{bmatrix} \begin{bmatrix} 18711.01871 \\ 52.4375 \end{bmatrix}$$

or $$b_0^* = \frac{18711.01871}{0.625} = 29937.62994 \tag{26.4}$$

and $$b_0^* - 25b_1^* = -2097.5$$

or $$b_1^* = \frac{-2097.5-b_0^*}{-25} = \frac{2097.5+29937.62994}{25}$$

or $$b_1^* = 1281.40525 \tag{26.5}$$

Substituting (26.4) and (26.5) into (26.3) yields

$$F_{r1}^*(s) = \frac{1281.40525s+29937.62994}{s^2+52.4375s+18711.01871} \tag{26.6}$$

Using the coefficients of $F_{r1}^*(s)$ in (26.6): $a_0^* = 18711.01871$, $a_1^* = 52.4375$, $b_1^* = 29937.62994$ as initial guesses, the nonlinear equations in (25) are solved by the Newton-Raphson method [15] and the following solutions are obtained at the 7th iteration with the error tolerance of 10^{-6}:

$$a_0 = 20917.459536$$
$$a_1 = 29.981293$$
$$b_1 = 957.260014$$

Since $b_0 = 1.6a_0$ as in (24.1) b_0 becomes

$$b_0 = 33467.93525$$

$\therefore$ $F_{s1}(s)$ the desired low-order stabilization filter in (23.1) is

$$F_{s1}(s) = \frac{957.260014s+33467.93525}{s^2+29.981293s+20917.459536} \tag{27}$$

The unit step response of the existing stabilized pitch control system in Eq. (1.5) and the redesigned pitch control system using $F_{s1}(s)$ in (27) and the $G_0(s)$ in (1.3) are shown in Fig. 4. The result is fairly satisfactory.

An alternate approach for redesigning the stabilization filter by direct simplification of the existing stabilization filter is proposed as follows:

As it is mentioned at the beginning of this chapter, the function of the stabilization filter is to convert the dominant data at $\omega = 0$, $\omega_{e\pi} = 1.9$ rad/sec. and $\omega_{ec} = 3.2$ rad/sec. of the original unstable system $G_0(s)$ in (3) to the assigned dominant data of $G_e(s)$ in (2). Taking advantage of this fact, we can directly apply the dominant-data matching method to fit a low-order stabilization filter that satisfies the specifications assigned in Eqn. (4). Let us assume that the desired low-order model of $F_{stab}(s)$ is

$$F_{s2}(s) = \frac{b_0+b_1s}{a_0+a_1s+s^2} \tag{28.1}$$

Applying the condition in (4.1) to $F_{s2}(s)$ in (28.1) yields

$$Re[F_{s2}(j0)] = \frac{b_0}{a_0} = 1.6$$

$$\therefore b_0 = 1.6a_0 \tag{28.2}$$

Substituting (28.2) into (28.1) gives

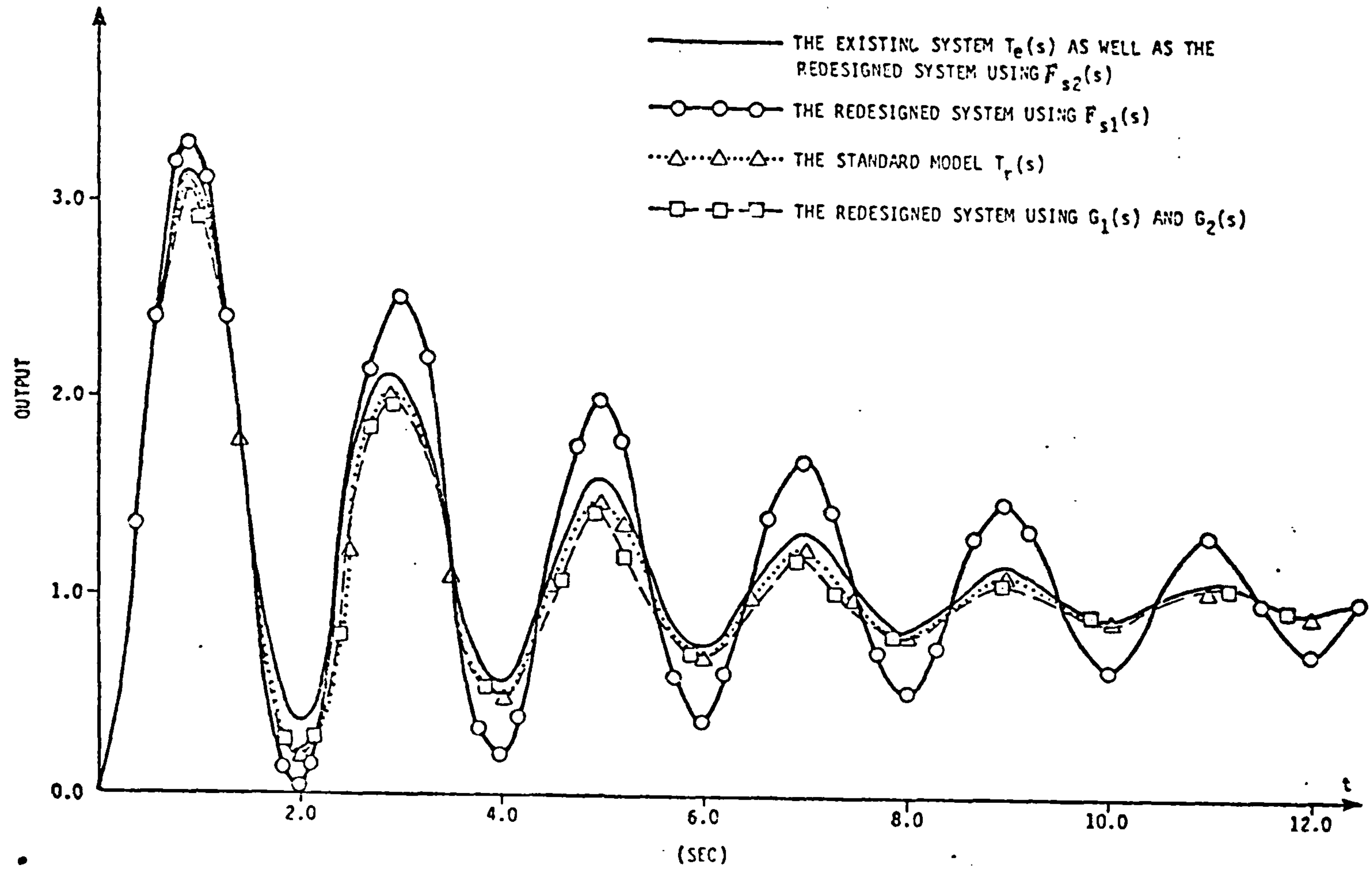

Figure 4. Time Responses of Various Models

$$F_{s2}(s) = \frac{1.6a_0+b_1 s}{a_0+a_1 s+s^2} \tag{28.3}$$

At $s = j\omega$

$$F_{s2}(j\omega) = \frac{1.6a_0+j\omega b_1}{(a_0-\omega^2)+j\omega a_1}$$

$$\therefore \quad |F_{s2}(j\omega)| = \frac{\sqrt{2.56a_0^2+\omega^2 b_1^2}}{\sqrt{(a_0-\omega^2)^2+\omega^2 a_1^2}} \tag{28.4}$$

and $$\underline{/F_{s2}(j\omega)} = \tan^{-1}\frac{\omega b_1}{1.6a_0} - \tan^{-1}\frac{\omega a_1}{a_0-\omega^2}$$

$$= \tan^{-1}\frac{\omega b_1(a_0-\omega^2)-1.6\omega a_0 a_1}{1.6a_0(a_0-\omega^2)+\omega^2 a_1 b_1} \tag{28.5}$$

At $s = j\omega_{e\pi} = j1.9$ the values of $|F_{s2}(j\omega)|$ and $\underline{/F_{s2}(j\omega)}$ in (28.4) and (28.5) respectively are matched to the corresponding values of $|F_{stab}(j1.9)|$ together and $\underline{/F_{stab}(j1.9)}$ in (4.3). Thus, we have

$$|F_{s2}(j1.9)| = \frac{\sqrt{2.56a_0^2+3.61b_1^2}}{\sqrt{(a_0-3.61)^2+3.61a_1^2}} = 1.605107127$$

or $$f_1(a_0,a_1,b_1) = 2.56a_0^2+3.61b_1^2-2.576368889[(a_0-3.61)^2$$

$$+ 3.61a_1^2] = 0 \tag{29.1}$$

and $$\underline{/F_{s2}(j1.9)} = \tan^{-1}\frac{1.9b_1(a_0-3.61)-3.04a_0a_1}{1.6a_0(a_0-3.61)+3.61a_1b_1} = 4.34591898^\circ$$

or $$f_2(a_0,a_1,b_1) = 1.9b_1(a_0-3.61)-3.04a_0a_1$$

$$- 0.0759963811[1.6a_0(a_0-3.61)+3.61a_1b_1]=0 \quad (29.2)$$

When $s = j\omega_{ec} = j3.2$ the value of $\underline{/F_{s2}(j\omega_{ec})}$ in (28.5) is compared with the value of $\underline{/F_{stab}(j\omega_{ec})}$ in (4.5). Thus, we have

$$\underline{/F_{s2}(j3.2)} = \tan^{-1} \frac{3.2b_1(a_0-10.24)-5.12a_0a_1}{1.6a_0(a_0-10.24)+10.24a_1b_1} = 7.293349493°$$

or $$f_3(a_0,a_1,b_1) = 3.2b_1(a_0-10.24)-5.12a_0a_1$$

$$- 0.1279849782[1.6a_0(a_0-10.24)+10.24a_1b_1]=0 \quad (29.3)$$

Using the initial guesses obtained in (26.6) the set of nonlinear equations in (29) is solved for the unknowns a_0, a_1 and b_1 by using the Newton-Raphson method. The solutions are obtained at the 9th iteration with the error tolerance of 10^{-6}. The solutions obtained are

$$a_0 = 13301.999297$$
$$a_1 = 3.318051$$
$$b_1 = 856.628596$$

Thus, the desired low-order model in (28.3) is

$$F_{s2}(s) = \frac{856.628596s+21283.19886}{s^2+3.318051s+13301.999297} \quad (30)$$

The unit-step response of the existing stabilized pitch control system $T_e(s)$ in (1.5) and the redesigned system that uses the low-order filter $F_{s2}(s)$ in (30) and $G_0(s)$ in (1.3) are shown in Fig. 4. The result is perfect. Comparing the unit-step response curves in Fig. 4, it is clear that as far as the performance of the entire pitch control system is concerned $F_{s2}(s)$ in (30) is a better filter than $F_{s1}(s)$ in (27). This implies that the existing stabilization filter $F_{stab}(s)$ in (1.2) might be overdesigned. Obviously, the implementation cost of the filter $F_{s2}(s)$ is less than that of $F_{stab}(s)$ in (1.2).

CHAPTER V

REDESIGN OF THE STABILIZATION FILTER BY AN ALGEBRAIC METHOD

In Chapter IV of this report the original fourth-order stabilization filter $F_{stab}(s)$ has been simplified to two second-order filters, $F_{s1}(s)$ and $F_{s2}(s)$, using the dominant-data matching method discussed in Chapter II. It is noticed that all three stabilization filters, the original as well as its simplified models consist of complex poles. It is also observed that all three filters mentioned above are placed in the feed forward loop and as a result the system becomes very sensitive to external disturbances. If alternate filters can be designed and placed in both feed forward and feedback loops, i) the designed filters may turn out to be simple transfer functions with positive real roots and because of this it may be possible to synthesize the filters using passive elements, and ii) the performances of the designed system can be greatly improved. The fact that the fixed configuration of the compensators in the feedback loop enables the designed system to be insensitive to the parameter variations and modeling errors will reduce the effects of external disturbances and improve the stability of the system. Thus the redesigned feedback system has all the advantages [14] of feedback control systems.

In this chapter the algebraic method proposed by Shieh [3] and Chen [4] is extended and modified to redesign the pitch control system The steps involved are summarized as follows.

Step 1. Assign the design goals using frequency-domain specifications and model a standard transfer function, known as the standard model, using the dominant data matching method discussed in Chapter II of this thesis.

Step 2. Expand the standard model obtained in Step 1 into a standard fraction expansion of the second Cauer form by performing repeated long divisions as shown in (13.0) to obtain the dominant quotients. Using these quotients obtain the matrix [H] in Equation (15).

Step 3. Assume the fixed configuration of compensators with unknown parameters and determine the overall transfer function of the system. Thus, the overall transfer function of the system will consist of the unknown parameters.

Step 4. Substitute the coefficients of the overall transfer function obtained in Step 3 into the vectors [a] and [b] in Equation (15) and expand the matrix equation (15) to obtain a set of equations.

Step 5. Solve the set of equations obtained in Step 4 to determine the unknown constants assigned in the compensators.

The designed system obtained by using the algebraic method has the exact dominant quotients of the standard model. In other words, the designed system is a good approximation of the standard model that has the exact assigned dominant data.

It is noticed that the original unstable system $G_0(s)$ in (1.3) is a high order transfer function with large coefficients. Therefore, in order to simplify the procedure, before proceeding to design the pitch control system by using the algebraic method, a reduced-order model of $G_0(s)$ is determined by using the dominant-data matching method.

The unstable transfer function $G_0(s)$ in (1.3) can be decomposed into a stable function and an unstable portion as follows:

$$G_0(s) = \frac{1}{s(s-2.921)} T_0(s) \tag{31.1}$$

where the stable portion $T_0(s)$ is

$$T_0(s) = \frac{324332.316(s+0.1933)(s+65)(s+1500)}{(s+3.175)(s+87.9\pm j95.5)(s+112.5)(s+1385)} \tag{31.2}$$

The pole at the origin and the unstable pole at $s = 2.921$ are considered to be the dominant poles of the system. Therefore, they are retained in the simplified model $G_0^*(s)$ of $G_0(s)$, or

$$G_0(s) \approx G_0^*(s) = \frac{1}{s(s-2.921)} T_0^*(s) \tag{31.3}$$

Where $T_0^*(s)$ is the reduced-order model of $T_0(s)$ obtained by using the dominant-data matching method. The frequency response data of $T_0(s)$ that are used as dominant data for the transfer function fitting are gain margin, phase margin, phase-crossover frequency, gain-crossover frequency, and the final value at $\omega = 0$. The $T_0^*(s)$ obtained is

$$T_0^*(s) = \frac{496.854897s^2+192897.961011s+37103.33375}{s^3+117.073733s^2+16552.300003s+50595.685093} \tag{31.4}$$

The $T_0^*(s)$ obtained is a low-order model of $T_0(s)$ with smaller coefficients. Thus, the design process can be greatly simplified.

Therefore $$G_0^*(s) = \frac{496.854897s^2+192897.961011s+37103.33375}{s^5+114.152733s^4+16210.32763s^3+2246.41679s^2-147789.9961s} \tag{31.5}$$

Following the steps proposed at the beginning of this chapter the first step to design a system by the algebraic method is to determine the standard model. In this case the standard model $T_r(s)$ has been determined earlier in Chapter III and is given in (9). Writing $T_r(s)$ once again and expanding it in a continued fraction expansion yields

$$T_r(s) = \frac{6.37807+20.55661s+0.243466s^2}{6.37807+10.46222s+1.259008s^2+s^3}$$

$$= \cfrac{1}{h_1+\cfrac{s}{h_2+\cfrac{s}{h_3+\cfrac{s}{h_4+\cfrac{s}{h_5+\cfrac{s}{h_6}}}}}}$$

where

$$\begin{aligned} h_1 &= 1 \\ h_2 &= -0.631845015 \\ h_3 &= -0.476189214 \\ h_4 &= 14.799589050 \\ h_5 &= -0.102867450 \\ h_6 &= -13.924278040 \end{aligned} \qquad (31.6)$$

In the next step a series compensator $G_1(s)$ and a parallel compensator $G_2(s)$ are assigned as shown in the block diagram of Fig. 5-1. $G_1(s)$

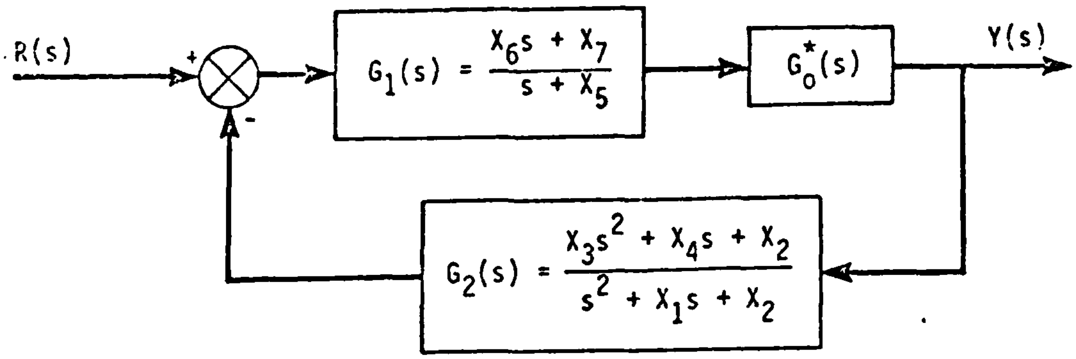

Fig. 5-1 The Block Diagram of a Redesigned System with Fixed Configuration Compensators

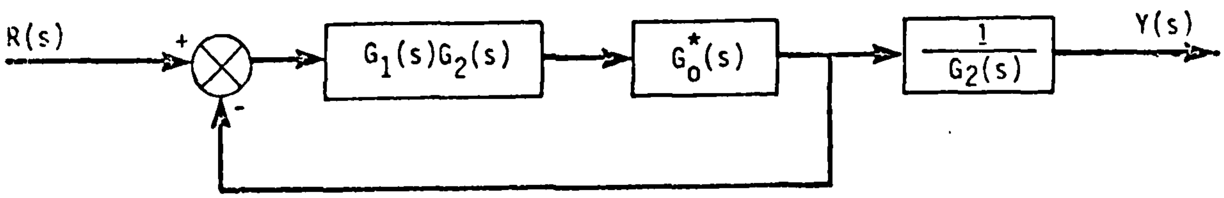

Fig. 5-2 The Modified Block Diagram of the Redesigned System

Figure 5 The Block Diagrams of the Redesigned System Using Algebraic Method and $G_2(s)$ are assumed with unknown parameters x_i, $i = 1,2,\ldots,7$ as

$$G_1(s) = \frac{x_6 s+x_7}{s+x_5} \tag{32.1}$$

and

$$G_2(s) = \frac{x_3 s^2+x_4 s+x_2}{s^2+x_1 s+x_2} \tag{32.2}$$

The overall transfer function $T_f(s)$ of the feedback system shown in Fig. 5-1 is

$$T_f(s) = \frac{b_0+b_1 s+ \ldots +b_7 s^7}{a_0+a_1 s+ \ldots +a_8 s^8} \tag{32.3}$$

where

$$\begin{aligned}
a_0 &= 37103.33375x_2x_7 \\
a_1 &= 192897.961011x_2x_7+37103.33375(x_2x_6+x_4x_7)-147789.9961x_2x_5 \\
a_2 &= 496.854897x_2x_7+192897.961011(x_2x_6+x_4x_7) \\
&+ 37103.33375(x_4x_6+x_3x_7)+2246.41679x_2x_5 \\
&- 147789.9961(x_2+x_1x_5) \\
a_3 &= 496.854897(x_2x_6+x_4x_7)+192897.961011(x_4x_6+x_3x_7) \\
&+ 37103.33375x_3x_6-147789.9961(x_1+x_5) \\
&+ 2246.41679(x_2+x_1x_5)+16210.32763x_2x_5 \\
a_4 &= 496.854897(x_4x_6+x_3x_7)+192897.961011x_3x_6-147789.9961 \\
&+ 2246.41679(x_1+x_5)+16210.32763(x_2+x_1x_5)+114.152733x_2x_5 \\
a_5 &= 496.854897x_3x_6+2246.41679+16210.32763(x_1+x_5) \qquad (32.4) \\
&+ 114.152733(x_2+x_1x_5)+x_2x_5 \\
a_6 &= 16210.32763+114.152733(x_1+x_5)+x_2+x_1x_5
\end{aligned}$$

$a_7 = 114.152733+x_1+x_5$

$a_8 = 1$

$b_0 = 37103.33375x_2x_7$

$b_1 = 192897.961011x_2x_7+37103.33375(x_2x_6+x_1x_7)$

$b_2 = 496.854897x_2x_7+192897.961011(x_2x_6+x_1x_7)$

$\quad + 37103.33375(x_1x_6+x_7)$

$b_3 = 496.854897(x_2x_6+x_1x_7)+192897.961011(x_1x_6+x_7)+37103.33375x_6$

$b_4 = 496.854897(x_1x_6+x_7)+192897.961011x_6$

$b_5 = 496.854897x_6$

$b_6 = 0$

$b_7 = 0$

In order to match the seven unknown parameters, x_i, $i = 1,2,\ldots,7$ in (32) for this type '1' system we need eight quotients h_i, $i=1,2,\ldots,8$ in (9). Therefore, the third order standard model in (9) with the quotients h_i given in (31.6) has to be amplified to a fourth-order system. This is done by inserting $h_7 = 100$ and $h_8 = 0.1$ as shown below.

$$T_r(s) = \frac{6.37807+20.55661s+0.243466s^2}{6.37807+10.46222s+1.259008s^2+s^3}$$

$$= \cfrac{1}{h_1+\cfrac{s}{h_2+\cfrac{s}{h_3+\cfrac{s}{h_4+\cfrac{s}{h_5+\cfrac{s}{h_6}}}}}}$$

$$
= \cfrac{1}{h_1 + \cfrac{s}{h_2 + \cfrac{s}{h_3 + \cfrac{s}{h_4 + \cfrac{s}{h_5 + \cfrac{s}{h_6 + \cfrac{s}{h_7 + \cfrac{s}{h_8}}}}}}}}
$$

$$
= T_a(s) = \frac{63.78098007+211.8989926s+22.87561717s^2+0.34346s^3}{63.78098007+110.9545225s+23.00917551s^2+11.30110515\ s^3+s^4}
$$

It [15] has been shown that $T_a(s)$ in (33) is a good approximation of the original standard model $T_r(s)$ in (9).

Substituting the h_i, $i = 1,\ldots,6$ in (31.6) including $h_7 = 100$ and $h_8 = 0.1$, the matrices $[H_1]$ and $[H_2]$ in (5) are obtained next.

$$H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.6318422 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0.30087727 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 14.167729 & 4.4528546 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1.1565286 & -0.45805621 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.24346000 & 20.55667 & 6.378098 & 0 & 0 \\ 0 & 0 & 0 & 1 & 23.189471 & 2055.2089 & 637.8098 & 0 \\ 0 & 0 & 0 & 0 & 0.34346 & 22.875617 & 211.89899 & 63.78098 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -0.63184224 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.52380951 & 0.30087727 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 7.1203141 & 4.4528546 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.42094152 & -0.4315751 & -0.45805 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1.259011 & 10.462223 & 6.378098 & 0 & 0 \\ 0 & 0 & 0 & 100.42094 & 125.4695 & 1045.7642 & 637.8098 & 0 \\ 0 & 0 & 0 & 1 & 11.301105 & 23.009176 & 110.95452 & 63.78098 \end{bmatrix}$$

Substituting the unknown constants a_i, $i = 0,1,\ldots,7$, b_i, $i = 0,1,\ldots,7$ and $[H_1]$ and $[H_2]$ obtained above into (15) yields a set of equations in terms of a_i and b_i as follows.

$$[b] = [H][a]$$
$$= [H_2]^{-1} [H_1] [a]$$

or $$[H_2][b] = [H_1][a]$$

or expanding the above matrix equation yields

$$f_1(a_i,b_i) = a_0-b_0 = 0$$

$$f_2(a_i,b_i) = b_0-0.6318422396(b_1-a_1) = 0$$

$$f_3(a_i,b_i) = a_1+0.30087727(a_2-b_2)-0.52380951b_1 = 0$$

$$f_4(a_i,b_i) = b_1+7.1203141b_2+4.4528546(b_3-a_3)-14.167729a_2 = 0$$

$$f_5(a_i,b_i) = a_2-1.1565286a_3-0.45805621(a_4-b_4)-0.42094152b_2 + 0.4315751b_3 = 0$$

$$f_6(a_i,b_i) = b_2+1.259011b_3+10.462223b_4+6.378098(b_5-a_5) - 0.24346000a_3-20.55667a_4 = 0$$

$$f_7(a_i,b_i) = a_3+23.189471a_4+2055.2089a_5+637.8098(a_6-b_6) - 100.42094b_3-125.4695b_4-1045.7642b_5 = 0$$

$$f_8(a_i,b_i) = b_3+11.301105b_4+23.009176b_5+110.95452b_6 + 63.78098(b_7-a_7)-0.34346a_4-22.875617a_5-211.89899a_6 = 0$$

where $i = 0,1,\ldots,7$.

Now, substituting the values of a_i and b_i in terms of x_i, $i = 1,2,\ldots,7$ from (32.4) yields a set of nonlinear equations shown below. It is noticed that as $a_0 = b_0$ the equation $f_1(a_i,b_i) = a_0-b_0 = 0$ gives no information. The rest of the equations are

$$f_1(x_1,\ldots,x_7) = x_2x_7+0.6318422396[x_7(x_4-x_1)-3.98319992x_2x_5] = 0 \tag{33.1}$$

$$\begin{aligned} f_2(x_1,\ldots,x_7) = {} & x_7(8.22822291x_2+8.522553136x_4-6.939879587x_1 \\ & + x_3-1)+x_2(1.582676549x_6-13.17807554x_5) \\ & + x_6(x_4-x_1)-3.983199922(x_2+x_1x_5) = 0 \end{aligned} \tag{33.2}$$

$$\begin{aligned} f_3(x_1,\ldots,x_7) = {} & x_2(-12.71361621x_6-x_5)+x_7(13.58355291x_1 \\ & + 1.820964317x_2-26.29716913x_4)+10.79844539(x_1x_6+x_7) \\ & - 13.31248704(x_4x_6+x_3x_7)+6.327224282(x_1+x_5) \\ & + 1.588477708x_6(1-x_3)+20.03527143(x_2+x_1x_5) = 0 \end{aligned} \tag{33.3}$$

$$\begin{aligned} f_4(x_1,\ldots,x_7) = {} & x_7(x_2+668.4670071x_4-281.48094x_1)+x_6(386.9860673x_2 \\ & + 362.767005-456.258273x_3)-647.2403649(x_4x_6+x_3x_7) \\ & - 57.53603068x_2x_5-548.5188427(x_2+x_1x_5) \\ & + 235.861385(x_1x_6+x_7)+235.2945185+590.5096275(x_1 \\ & + x_5) = 0 \end{aligned} \tag{33.4}$$

$$\begin{aligned} f_5(x_1,\ldots,x_7) = {} & 2357.408023(x_1x_6+x_7)+x_6(1598.839931x_2+17096.15228 \\ & - 32881.95043x_3)+x_7(4.16745091x_2+1599.839931x_1-x_4) \\ & - 472.6735322(x_4x_6+x_3x_7)+24996.98242-939.0287936(x_1 \\ & + x_5)-2765.323026(x_2+x_1x_5)-52.07771943x_2x_5 = 0 \end{aligned} \tag{33.5}$$

$$f_6(x_1,\ldots,x_7) = x_6(-99.4209415x_2+11132.91981x_3-57256.87822)$$
$$+ x_7(x_4-100.4209415x_1)+411.4274907(x_4x_6+x_3x_7)$$
$$+ 67006.93001(x_1+x_5)+1234.567433(x_2+x_1x_5)$$
$$+ 42.69011171x_2x_5 + 23203.53455-39112.69694(x_1x_6$$
$$+ x_7) = 0 \qquad (33.6)$$

$$f_7(x_1,\ldots,x_7) = 496.854897(x_2x_6+x_1x_7)+198512.9704(x_1x_6+x_7)$$
$$+ 2228495.695x_6-170.6497831(x_4x_6+x_3x_7)$$
$$- 77618.59617x_3x_6-3442861.087-395845.4335(x_1+x_5)$$
$$- 8390.812346(x_2+x_1x_5)-62.08251489x_2x_5 = 0 \qquad (33.7)$$

Equation (33) is a set of high order nonlinear simultaneous equations which is very difficult to solve. However, with proper initial guesses the Newton-Raphson [15] method can be applied to solve it. Therefore, the problem lies in finding an appropriate set of initial guesses. In this case, the following method is suggested for estimating the initial guesses.

As mentioned earlier, the block diagram of the structure of the desired fixed configuration control system is shown in Fig. 5-1. Without affecting the overall transfer function of the system, this structure can be modified into a form as shown in Fig. 5-2. The overall transfer function of this structure is

$$T_1(s) = T_2(s)\,\frac{1}{G_2(s)} \qquad (34.1)$$

where

$$T_2(s) = \frac{G_1(s)G_2(s)G_0^*(s)}{1+G_1(s)G_2(s)G_0^*(s)}$$

Where $G_0^*(s)$, the approximate transfer function of $G_0(s)$, is given in (31.5).

The purpose is to determine $G_1(s)$ and $G_2(s)$ such that the response of $T_1(s)$ becomes close to that of the standard model $T_r(s)$ in (9). Replacing the series compensator $G_1(s)G_2(s)$ in Fig. (5-2) by the designed stabilization filter $F_{s2}(s)$ in (30) the resulting transfer function $T_1(s)$ in (34.1) is equated to the standard model $T_r(s)$ in (9) as follows.

$$T_1(s) \underset{\sim\sim\sim}{\text{force}} T_r(s)$$

or $$T_2(s)\frac{1}{G_2(s)} = T_r(s)$$

or $$G_2(s) = \frac{T_2(s)}{T_r(s)}$$

or $$G_2^*(s) \simeq G_2(s) = \frac{1}{T_r(s)}\left[\frac{G_1(s)G_2(s)G_0^*(s)}{1+G_1(s)G_2(s)G_0^*(s)}\right]$$

$$= \frac{1}{T_r(s)}\left[\frac{F_{s2}(s)G_0^*(s)}{1+F_{s2}(s)G_0^*(s)}\right]$$

$$= \frac{1}{T_r(s)}\left[\frac{789677630.6+4137269440s}{789677630.6+2171367017s}\right.$$

$$\frac{+175816571.1s^2+425620.1128s^3}{+205208030.9s^2+215915050.5s^3}$$

$$\frac{}{+154492.684s^4+29891.09151s^5}$$

$$\left.\frac{}{+117.470784s^6+s^7}\right] \tag{34.2}$$

Substituting $T_r(s)$ in (9) into (34.2) and simplifying, the appropriate transfer function $G_2^*(s)$ of $G_2(s)$ is obtained.

$$G_2^*(s) = \frac{5.036619205\times10^9+3.46495752\times10^{10}s+4.540060393\times10^{10}s^2}{5.036619205\times10^9+3.008227329\times10^{10}s+4.613716606\times10^{10}s^2}$$

$$\frac{+7.840679235\times10^9s^3+4.363076841\times10^9s^4+1.763524302\times10^8s^5}{+6.124169121\times10^9s^3+4.498497844\times10^9s^4+8.512494768\times10^7s^5}$$

$$\frac{+4.256201128\times10^5s^6}{+9.985459768\times10^5s^6+9.698650697\times10^3s^7+49.1568119s^8+0.243466s^9} \qquad (34.3)$$

A set of dominant quotients h_i of $G_2^*(s)$, given below, are determined by expanding (34.3) into a continued fraction of the second Cauer form

$$\begin{aligned}
h_1 &= 1\\
h_2 &= -1.102755917\\
h_3 &= -0.1287948973\\
h_4 &= 5.593229805\\
h_5 &= 0.1338916858\\
h_6 &= \ldots\\
&\vdots\\
h_{18} &= \ldots
\end{aligned} \qquad (34.4)$$

Substituting the first five quotients $h_1,h_2,\ldots,h_5$ into (13.5) gives a second order approximate model $G_2^{**}(s)$ of the approximate parallel filter $G_2^*(s)$ in (34.3) as

$$G^{**}(s) = \frac{0.994929057s^2+0.7394973923s^2+0.1058245527}{s^2+0.643533679s+0.1058245527} \quad (34.5)$$

$G_2^{**}(s)$, the approximate model of $G_2^{*}(s)$, is also an approximate model of $G_2(s)$ in (32.2).

The approximate model $G_1^{*}(s)$ of the series compensator $G_1(s)$ in Fig. 5-1 can be obtained as follows

$$G_1^{*}(s) \simeq \frac{F_{s2}(s)}{G_2^{**}(s)} = \frac{856.628596s^3+21834.46821s^2+13787.1076s}{0.994929057s^4+4.040722745s^3} \cdot$$

$$\frac{+2252.284999}{+13237.10512s^2+9837.144919s+1407.678125} \quad (35.1)$$

To obtain a set of dominant quotients Equation (35.1) is expanded into a continued fraction of the second Cauer form. Some of the quotients obtained are

$$\begin{aligned} h_1 &= 0.625 \\ h_2 &= 1.845828612 \\ h_3 &= 0.0839039052 \\ h_4 &= \ldots \\ &\vdots \\ h_8 &= \ldots \end{aligned} \quad (35.2)$$

The first three quotients h_1, h_2, h_3 are substituted into (13.3), which gives $G_1^{**}(s)$, an approximate model of $G_1^{*}(s)$ as well as of $G_1(s)$ in (32.1). $G_1^{**}(s)$ obtained is

$$G_1^{**}(s) = \frac{1.410628426s+0.2184671685}{s+0.1365419803} \tag{35.2}$$

Comparing (32.2) with (34.5) and (32.1) with (35.2) we have a set of initial guesses to solve the set of high order nonlinear equations in (33). Thus, the set of initial guesses is

$$
\begin{aligned}
x_1^* &= 0.643533679 \\
x_2^* &= 0.1058245527 \\
x_3^* &= 0.994929057 \\
x_4^* &= 0.7394973923 \\
x_5^* &= 0.1365419803 \\
x_6^* &= 1.410628426
\end{aligned}
\tag{36}
$$

and

$$x_7^* = 0.2184671685$$

Using these initial guesses the Newton-Raphson method [15] is applied to solve the nonlinear equations in (33). It is found that the Newton-Raphson method converges to the desired solutions, given below, at the 14th iteration with the error tolerance of 10^{-6}. The solutions are

$$
\begin{aligned}
x_1 &= 0.503850 \\
x_2 &= 0.059928 \\
x_3 &= 1.051503 \\
x_4 &= 0.580016
\end{aligned}
$$

$x_5 = 4.831826$

$x_6 = 1.885577$

$x_7 = 6.744450$

Therefore, the desired compensators $G_1(s)$ and $G_2(s)$ are

$$G_1(s) = \frac{1.885577s+6.744450}{s+4.831826} = \frac{1.885577(s+3.57688)}{s+4.831826} \quad (37.1)$$

and

$$G_2(s) = \frac{1.051503s^2+0.580016s+0.059928}{s^2+0.503850s+0.059928} = \frac{1.051503(s+0.13769)(s+0.41391)}{(s+0.19244)(s+0.311405)} \quad (37.2)$$

The unit step response curves of the existing stabilized system $T_e(s)$ in (1.5) and the redesigned system using the compensators $G_1(s)$ and $G_2(s)$ in (37), and $G_0(s)$ in (1.3), are compared in Fig. 4. The result is satisfactory.

It is interesting to note that $G_1(s)$ and $G_2(s)$ in Eq. (37) are positive real functions with positive real poles and zeros, which makes it possible to realize the compensators $G_1(s)$ and $G_2(s)$ using passive elements, whereas, the existing stabilization filter $F_{stab}(s)$ is a non-positive real function and it is realized by using active elements.

CHAPTER VI

CONCLUSION

The existing stabilized pitch control system has been redesigned by redesigning the existing stabilization filter. Two computer-oriented methods, a dominant data matching method and an algebraic method, have been presented to redesign the existing stabilization filter. Thus, various low-order stabilization filters have been obtained. As a result, the implementation cost of the missile system is reduced.

The proposed dominant-data matching method can be used for various purposes. For example, when the specifications or the design goals of a control system are given, the proposed method can be used to obtain a standard transfer function, which significantly simplifies the design process in the frequency domain. When a high-order transfer function is given, various low-order models can be obtained with the help of the dominant-data matching method. The method can be used in the problems of identification as well. The great advantage of this method is that the transfer functions obtained by using this method have the exact assigned frequency-domain specifications.

The algebraic method has been applied to achieve the advantages of the feedback control system so that the performances of the redesigned pitch control system can be greatly improved.

The application of the dominant-data matching method always gives rise to a set of nonlinear equations which can be solved if a set of proper initial guesses is known. In this connection, various methods have been discussed for estimating a set of proper initial guesses.

Finally, it is important to mention that the proposed computer-aided design methods can be used to design general control systems.

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A matrix in the block Schwarz form and the stability of matrix polynomials

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A matrix which consists of block elements is established in the block Schwarz form via a linear transformation. The transformation matrix constructed by Chen and Chu is modified and extended for transforming the block companion form to the block Schwarz form. A sufficient condition is derived for determining the stability of a multivariable system whose characteristics are expressed by a matrix polynomial. The matrix polynomial may or may not be symmetric.

1. Introduction

The properties and applications of the Schwarz matrix, which has scalar elements, has been investigated by various authors (Schwarz 1956, Parks 1963, Wall 1948, Anderson *et al.* 1976, Barnett and Storey 1970), and the transformation matrix, which relates various matrix forms and the Schwarz form, has also been established by numerous authors (Butchart 1965, Chen and Chu 1966, 1967, Barnet and Storey 1967, Loo 1968, Power 1969, Datta 1974, Sarma *et al.* 1968). Chen and Chu (1966, 1967) constructed a transformation matrix which links the Schwarz form and the companion form by using the Routh array elements (Routh 1877). However, all existing methods (Schwarz 1956, Parks 1963, Wall 1948, Anderson *et al.* 1976, Barnett and Storey 1967, 1970, Butchart 1965, Chen and Chu 1966, 1967, Loo 1968, Power 1969, Datta 1974, Sarma *et al.* 1968) deal only with the system matrix which has scalar elements but not block elements. In this paper a matrix which consists of block elements is constructed in the block Schwarz form and a linear transformation matrix which consists of block elements is established to transform the matrix in the block companion form (Shieh 1975) (the block phase variable form) to the block Schwarz form. A sufficient condition is then derived to determine the stability of a multivariable system whose characteristics are described by a matrix polynomial (Shieh 1975, Shieh *et al.* 1976). The matrix polynomial may or may not be symmetric.

2. A transformation for a matrix in the Schwarz block form

Consider a set of linear, time-invariant, ordinary differential equations in the differential matrix polynomial form

$$\sum_{i=1}^{n+1} B_i D^{i-1} X(t) = [0], \quad B_{n+1} = I \tag{1 a}$$

$$D^{i-1} X(0) = [\alpha_{i-1}], \quad i = 1, 2, \ldots, n \tag{1 b}$$

where $X(t)$ is the m-dimensional state of the system. The B_i are $m \times m$ real constant matrices and the differential operator D is $D = d/dt$. The matrix I

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is an identity matrix and [0] is a null matrix. The corresponding state equation of eqn. (1) in the block companion form is

$$[\dot{x}]=[B][x] \tag{2 a}$$

$$[x(0)]=[\alpha] \tag{2 b}$$

where

$$[B]=\begin{bmatrix} 0 & I & 0 & 0 & \cdot & 0 \\ 0 & 0 & I & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & I \\ -B_1 & -B_2 & -B_3 & -B_4 & \cdot & -B_n \end{bmatrix}$$

The dimensions of the matrix $[B]$, each block element in the $[B]$, and the state vector $[x]$ are $(n \times m) \times (n \times m)$, $m \times m$, and $(n \times m) \times 1$, respectively. The $[B]$ is the matrix in the block companion form or the block phase variable form (Shieh 1975).

Equation (2) can be transformed into the block Schwarz form by using the following linear transformation $[K_1]$:

$$[y]=[K_1][x] \tag{3}$$

and

$$\begin{aligned}[\dot{y}]&=[K_1][B][K_1]^{-1}[y] \\ &=[A][y]\end{aligned} \tag{4}$$

where

$$[K_1]=\begin{bmatrix} I & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ C_{n-1,1}^{-1}C_{n-1,2} & \cdot & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdot & 0 & I & 0 & 0 & 0 & 0 & 0 \\ C_{n-3,1}^{-1}C_{n-3,3} & \cdot & C_{61}^{-1}C_{62} & 0 & I & 0 & 0 & 0 & 0 \\ 0 & \cdot & 0 & C_{51}^{-1}C_{52} & 0 & I & 0 & 0 & 0 \\ C_{n-5,1}^{-1}C_{n-5,4} & \cdot & C_{41}^{-1}C_{43} & 0 & C_{41}^{-1}C_{42} & 0 & I & 0 & 0 \\ 0 & \cdot & 0 & C_{31}^{-1}C_{33} & 0 & C_{31}^{-1}C_{32} & 0 & I & 0 \\ \cdot & \cdot & C_{21}^{-1}C_{24} & 0 & C_{21}^{-1}C_{23} & 0 & C_{21}^{-1}C_{22} & 0 & I \end{bmatrix}$$

and

$$[A]=\begin{bmatrix} 0 & I & 0 & 0 & \cdot & 0 & 0 \\ -A_1 & 0 & I & 0 & \cdot & 0 & 0 \\ 0 & -A_2 & 0 & I & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & I \\ 0 & 0 & 0 & 0 & \cdot & -A_{n-1} & -A_n \end{bmatrix}$$

The dimension of each block element in the matrix $[A]$ and the matrix $[K_1]$ is $m \times m$. The $[A]$ is the matrix in the block Schwarz form. The linear transformation matrix $[K_1]$, which relates the coordinates $[x]$ and $[y]$ in eqns. (2) and (4), is constructed by following the approach proposed by Chen and Chu (1966). The block elements $C_{i,j}$ having dimension $m \times m$ in eqn. (3) can be obtained from the following matrix Routh algorithm and the matrix Routh array (Shieh and Gaudiano 1974, Shieh *et al.* 1976).

Before performing the matrix Routh array we define $l=(n/2)+1$ if n is an even number; otherwise, $l=n+1/2$, and the double subscripted block elements $C_{1,j}$ and $C_{2,j}$ become:

$$\left.\begin{array}{ll} C_{1,j}=B_{n+3-2j}, & j=1,2,3,\ldots,l \\ C_{2,j}=B_{n+2-2j}, & j=1,2,3,\ldots,l \\ C_{11}=I & \end{array}\right\} \qquad (5\,a)$$

The B_i are the $m \times m$ real constant matrices shown in eqn. (1). The matrix Routh array and the matrix Routh algorithm are

$$\left.\begin{array}{ll} H_1=C_{11}C_{21}^{-1} & \angle\begin{array}{cccccc} C_{11} & C_{12} & C_{13} & C_{14} & \cdots \\ C_{21} & C_{22} & C_{23} & C_{24} & \cdots \\ C_{31} & C_{32} & C_{33} & \cdot & \cdots \\ C_{41} & C_{42} & C_{43} & \cdot & \cdots \\ C_{51} & C_{52} & \cdot & \cdot & \cdots \\ C_{61} & C_{62} & \cdot & \cdot & \cdots \\ C_{71} & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \end{array} \\ H_2=C_{21}C_{31}^{-1} & \\ H_3=C_{31}C_{41}^{-1} & \\ H_4=C_{41}C_{51}^{-1} & \\ H_5=C_{51}C_{61}^{-1} & \\ H_6=C_{61}C_{71}^{-1} & \\ H_n=C_{n,1}C_{n+1,1}^{-1} & \angle\begin{array}{c} C_{n,1} \\ C_{n+1,1} \end{array} \end{array}\right\} \qquad (5\,b)$$

where

$$C_{i,j}=C_{i-2,j+1}-H_{i-2}C_{i-1,j+1}, \quad j=1,2,\ldots, \quad i=3,4,\ldots$$

$$H_i=C_{i,1}(C_{i+1,1})^{-1}, \quad i=1,\ldots,n$$

$$\det(C_{i+1,1}) \neq 0$$

The matrices H_i in eqn. (5 *b*) are called the matrix quotients. Performing a new linear transformation

$$[z]=[K_2][y] \qquad (6)$$

2 F 2

on eqn. (4) yields an alternative matrix $[F]$ in the block Schwarz form as follows :

$$\begin{aligned}[\dot{z}] &= [K_2][A][K_2]^{-1}[z] \\ &= [F][z] \end{aligned} \tag{7 a}$$

where

$$[K_2] = \begin{bmatrix} C_{n,1} & 0 & \cdot & 0 & 0 \\ 0 & C_{n-1,1} & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & C_{21} & 0 \\ 0 & 0 & \cdot & 0 & C_{11} \end{bmatrix} \tag{7 b}$$

and

$$[F] = \begin{bmatrix} 0 & H_{n-1}^{-1} & 0 & \cdot & 0 & 0 & 0 & 0 \\ -H_n^{-1} & 0 & H_{n-2}^{-1} & \cdot & 0 & 0 & 0 & 0 \\ 0 & -H_{n-1}^{-1} & 0 & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & H_3^{-1} & 0 & 0 \\ 0 & 0 & 0 & \cdot & -H_4^{-1} & 0 & H_2^{-1} & 0 \\ 0 & 0 & 0 & \cdot & 0 & -H_3^{-1} & 0 & H_1^{-1} \\ 0 & 0 & 0 & \cdot & 0 & 0 & -H_2^{-1} & -H_1^{-1} \end{bmatrix} \tag{7 c}$$

The $[K_2]$ is a block diagonal matrix having the diagonal block elements obtained from the block elements $C_{i,1}$, $i = 1, 2, \ldots$, which are in the first column of the matrix Routh array in eqn. (5 b), while the matrix $[F]$ is the required matrix in the block Schwarz form which can be constructed by using the matrix quotients H_i, $i = 1, 2, \ldots$, obtained from the same matrix Routh array. A similar matrix (Schwarz and Friedland 1965), which was formulated in the Schwarz form but not in the block Schwarz form, was used to prove the stability of a linear system by Parks (1963).

The linear transformation matrix $[K]$, which links the coordinates $[x]$ in the block companion form and the coordinates $[z]$ in the block Schwarz form, is

$$\begin{aligned}[z] &= [K_2][K_1][x] \\ &= [K][x] \end{aligned} \tag{8 a}$$

where

$$
[K]=\begin{bmatrix}
H_nC_{n+1,1} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
H_{n-2}C_{n-1,2} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cdots & H_6C_{71} & 0 & 0 & 0 & 0 & 0 \\
H_{n-4}C_{n-3,3} & \cdots & 0 & H_5C_{61} & 0 & 0 & 0 & 0 \\
0 & \cdots & H_4C_{52} & 0 & H_4C_{51} & 0 & 0 & 0 \\
H_{n-6}C_{n-5,4} & \cdots & 0 & H_3C_{42} & 0 & H_3C_{41} & 0 & 0 \\
0 & \cdots & H_2C_{33} & 0 & H_2C_{32} & 0 & H_2C_{31} & 0 \\
\cdot & \cdots & 0 & H_1C_{23} & 0 & H_1C_{22} & 0 & H_1C_{21}
\end{bmatrix} \qquad (8\,b)
$$

The matrix $[K]$ is a block triangular matrix. All the block elements in eqn. (8 *b*) can be directly obtained from the matrix Routh array in eqn. (5 *b*). For example, the block elements in the main diagonal, which are shown by the first dotted diagonal line, are obtained by the respective products of the matrix quotients H_i and block elements $C_{i,1}$ in the first column of the matrix Routh array. The block elements of the first lower diagonal in the $[K]$ are null matrices, and the block elements of the second lower diagonal in the $[K]$, which are shown by the second dotted diagonal line, are determined by the respective products of the matrix quotients H_i and the block elements $C_{i,2}$ in the second column of the same matrix Routh array, etc. The sizes of the matrices $[F]$ and $[K]$ are determined by the degree of the matrix polynomial and the order of the matrix coefficients in eqn. (1). For instance, when the degree of a matrix polynomial is $n=4$ and the dimension of each matrix coefficient is m, the corresponding $4m \times 4m$ square matrices $[F]$ and $[K]$ are taken from the right-hand side lower corner of the matrices $[F]$ and $[K]$ in eqns. (7 *c*) and (8 *b*).

3. A sufficient condition for the stability of a matrix polynomial

In a single variable system the Routh criterion (Routh 1877) is applied to the characteristic polynomial of a linear system for determining the absolute stability. In other words, the scalar polynomial in the form of eqn. (1) is arranged in the Routh array in eqn. (5 *b*), then the Routh criterion is applied. If the scalars $C_{i,1}$ in the first column of the Routh array have no sign changes or all elements $C_{i,1}$, $i=1, 2, \ldots$, are positive real, then the system is asymptotically stable. Since the Routh algorithm and the Routh array have been successfully extended to the matrix Routh algorithm and the matrix Routh array (Shieh and Gaudiano 1974, Shieh *et al.* 1976), also a positive definite matrix (Bellman 1970) is commonly considered as a natural generalization of a positive number, it is interesting to ask whether or not a multivariable system whose characteristic matrix polynomial shown in eqn. (1) is asymptotically stable if the block elements $C_{i,1}$, $i = 1, 2, \ldots$, in the first column of the matrix Routh array in eqn. (5 b) are positive definite matrices. In other words, can we directly extend the Routh criterion (Routh 1877) to the matrix Routh criterion? This paper will partially answer this question.

Because the stability of a system is invariant under the transpose operation of the system matrix, we consider the following system :

$$\begin{aligned}[\dot{q}] &= [F^{\mathrm{T}}][q] \\ &= [G][q] \end{aligned} \tag{9}$$

The matrix $[F]$ in eqn. (9) is defined in eqn. (7) and the transpose of the matrix $[F]$ is defined as $[G]$. If the matrix quotients H_i in eqn. (5 b) are positive-definite symmetric and real matrices, then we can consider the following quadratic equation (Kalman and Bertram 1960, Barnett 1971) :

$$v = [q^{\mathrm{T}}][P][q] \tag{10 a}$$

where

$$[P] = \begin{bmatrix} H_n & 0 & \cdot & 0 & 0 \\ 0 & H_{n-1} & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & H_2 & 0 \\ 0 & 0 & \cdot & 0 & H_1 \end{bmatrix}$$

The derivative function $\dot{v}$ is

$$\begin{aligned}\dot{v} &= [q^{\mathrm{T}}][PG + G^{\mathrm{T}}P][q] \\ &= -[q^{\mathrm{T}}][Q][q] \end{aligned} \tag{10 b}$$

where

$$[P][G] = \begin{bmatrix} 0 & -I & 0 & \cdot & 0 & 0 \\ I & 0 & -I & \cdot & 0 & 0 \\ 0 & I & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & -I \\ 0 & 0 & 0 & \cdot & I & -I \end{bmatrix}, \quad [Q] = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 2I \end{bmatrix}$$

The v function in eqn. (10 a) is in a positive-definite quadratic form and the $\dot{v}$ function in eqn. (10 b) is in a negative-semidefinite form. Therefore the system in eqn. (9) or in eqn. (1) is asymptotically stable. From the above derivation we obtain the sufficient condition that a multivariable system, whose characteristic matrix polynomial has the form shown in eqn. (1), is asymptotically stable if the matrix quotients H_i in eqn. (5 b) are real symmetric positive-definite matrices. From eqns. (2 a) and (7 c) it can be observed that the $B_n (= H_1^{-1} = C_{21})$ must be symmetric and positive-definite for the sufficient condition to apply. It is also noted that this sufficient condition deals only with H_i and not $C_{i,j}$. This implies that, if a system is asymptotically stable, the block elements $C_{i,1}$, $i = 1, 2, \ldots$, in the first column of the matrix Routh array and the B_i in eqn. (1) are not necessarily symmetric and positive-definite matrices. In other words, a positive-definite matrix is not necessarily a natural generalization of a positive number, and the necessary and sufficient condition of the Routh criterion (Routh 1877) cannot be completely extended to the matrix Routh criterion for general cases.

On the other hand, the necessary conditions for asymptotic stability due to the Routh criterion (Routh 1877) can be partially extended to the case of matrix polynomials. The necessary conditions are described as follows:

(i) The determinant of B_1 in eqn. (1) is non-zero.

(ii) The determinants of B_{n+1} and B_1 in eqn. (1) have the same sign if the determinant of $B_{n+1}(=C_{11})$ is non-zero.

These conditions can be easily verified from the basic laws of algebra. Thus, in this paper, we have partially extended the Routh criterion (Routh 1877) to the matrix Routh criterion for determining the asymptotic stability of a class of matrix polynomials.

Sometimes in applying the approach proposed in this paper difficulties may be encountered in calculating the matrix quotients H_i in eqn. (5 *b*). This implies that if any $C_{i,1}$ in eqn. (5 *b*) is singular, then the H_i cannot be obtained to determine the stability of a matrix polynomial. This limitation can be remedied by multiplying the original matrix polynomial, defined as $T(s)$, by a polynomial matrix defined as $E(s)$, where the roots of the determinant $E(s)$ have negative real parts. Then we apply the matrix Routh procedure to the modified matrix polynomial $T(s)E(s)$ or $E(s)T(s)$. It is noted that the stability of the original system is reserved because the stability of a system is invariant under this modification. An alternative way is to perform transformation $s \to 1/s$ to the $T(s)$ and then applying the matrix Routh procedure to the modified matrix polynomial defined as $T_1(s)(=T(s)|_{s\to 1/s})$. In other words, the $C_{1,j}$ and $C_{2,j}$ in eqn. (5 *a*) are rewritten as follows:

$$C_{1,j} = B_{2j-1} \quad \text{for } j = 1, 2, 3, \ldots$$
$$C_{2,j} = B_{2j} \quad \text{for } j = 1, 2, 3, \ldots$$

Again, the stability of the original system is invariant to this modification and the numerically ill-conditioned problem (i.e. $C_{i,1}$ is singular) can be solved. Examples are established in this paper to show the procedure.

4. Illustrative examples

4.1. *Example* 1

Consider the following differential matrix polynomial:

$$\sum_{i=1}^{n+1=5} B_i D^{i-1} X(t) = [0] \tag{11}$$

or

$$B_5 \overset{(4)}{X}(t) + B_4 \overset{(3)}{X}(t) + B_3 \ddot{X}(t) + B_2 \dot{X}(t) + B_1 X(t)$$
$$= C_{11} \overset{(4)}{X}(t) + C_{21} \overset{(3)}{X}(t) + C_{12} \ddot{X}(t) + C_{22} \dot{X}(t) + C_{13} X(t) = [0]$$

where

$$C_{11}=B_5=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

$$C_{12}=B_3=\begin{pmatrix}-37\cdot05 & -78\cdot8\\ 33 & 65\end{pmatrix}\quad C_{13}=B_1=\begin{pmatrix}-10\cdot5 & -23\\ -0\cdot1 & -0\cdot6\end{pmatrix},$$

$$C_{21}=B_4=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\qquad C_{22}=B_2=\begin{pmatrix}-43\cdot1 & -94\cdot6\\ -6\cdot05 & -16\cdot3\end{pmatrix}$$

A matrix in the block Schwarz form and the linear transformation matrix which transforms the block companion form to the block Schwarz form are required to be constructed, and the stability of the system is of interest.

Arranging the matrices B_i in eqn. (11) in the matrix Routh array in eqn. (5 b) results in the following :

$$\left.\begin{array}{ll}
 & C_{11}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\quad C_{12}=\begin{pmatrix}-37\cdot05 & -78\cdot8\\ 33 & 65\end{pmatrix}\\
H_1=\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix} & \qquad\qquad\qquad\qquad C_{13}=\begin{pmatrix}-10\cdot5 & -23\\ -0\cdot1 & -0\cdot6\end{pmatrix}\\
 & C_{21}=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\quad C_{22}=\begin{pmatrix}-43\cdot1 & -94\cdot6\\ -6\cdot05 & -16\cdot3\end{pmatrix}\\
H_2=\begin{pmatrix}4 & 1\\ 1 & 0\cdot5\end{pmatrix} & \\
 & C_{31}=\begin{pmatrix}0 & -0\cdot5\\ 2 & 3\end{pmatrix}\quad C_{32}=\begin{pmatrix}-10\cdot5 & -23\\ -0\cdot1 & -0\cdot6\end{pmatrix}\\
H_3=\begin{pmatrix}1\cdot125 & 0\cdot25\\ 0\cdot25 & 0\cdot5\end{pmatrix} & \\
 & C_{41}=\begin{pmatrix}-1 & -2\\ 4\cdot5 & 7\end{pmatrix}\\
H_4=\begin{pmatrix}0\cdot1 & -0\cdot5\\ -0\cdot5 & 7\cdot5\end{pmatrix} & \\
 & C_{51}=\begin{pmatrix}-10\cdot5 & -23\\ -0\cdot1 & -0\cdot6\end{pmatrix}
\end{array}\right\}\qquad(12)$$

The matrix quotients H_i, $i=1, 2, \ldots, 4$, in eqn. (12) are positive-definite symmetric real matrices ; therefore the system is asymptotically stable. It is noted that the block elements $C_{i,1}$, $i=1, \ldots, 5$ in the first column of the matrix Routh array in eqn. (12) are not all positive-definite symmetric real matrices. The state equation in the block companion form is

$$[\dot{x}]=[B][x] \tag{13}$$

where

$$[B]=\begin{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 10{\cdot}5 & 23 \\ 0{\cdot}1 & 0{\cdot}6 \end{pmatrix} & \begin{pmatrix} 43{\cdot}1 & 94{\cdot}6 \\ 6{\cdot}05 & 16{\cdot}3 \end{pmatrix} & \begin{pmatrix} 37{\cdot}05 & 78{\cdot}8 \\ -33 & -65 \end{pmatrix} & \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix} \end{bmatrix}$$

and the state equation in the block Schwarz form is

$$[\dot{z}]=[F][z] \tag{14 a}$$

where

$$[F]=\begin{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & -0{\cdot}5 \\ -0{\cdot}5 & 2{\cdot}25 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -15 & -1 \\ -1 & -0{\cdot}2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0{\cdot}5 & -1 \\ -1 & 4 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0{\cdot}5 \\ 0{\cdot}5 & -2{\cdot}25 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -0{\cdot}5 & 1 \\ 1 & -4 \end{pmatrix} & \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix} \end{bmatrix}$$

It is interesting to note that the characteristic equation

$$|sI-B|=|sI-F|=0$$

and the roots which have negative real parts are, respectively :

$$s^8+3s^7+28\cdot 95s^6+79\cdot 35s^5+206s^4+458\cdot 875s^3 + 221\cdot 05s^2+48\cdot 4s+4=0 \qquad (14\,b)$$

and

$$\left.\begin{array}{l} -0\cdot 0239155 \pm j4\cdot 27199 \\ -0\cdot 0784809 \pm j2\cdot 95637 \\ -0\cdot 189163 \quad \pm j0\cdot 165319 \\ -0\cdot 177194 \\ -2\cdot 23969 \end{array}\right\} \qquad (14\,c)$$

The linear transformation matrix between the $[x]$ and $[z]$ coordinates is

$$[z]=[K][x] \qquad (15)$$

where

$$[K]=\begin{bmatrix} \begin{pmatrix} -1 & -2 \\ 4\cdot 5 & 7 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -0\cdot 5 \\ 2 & 3 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -42\cdot 1 & -92\cdot 6 \\ -10\cdot 55 & -23\cdot 3 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -37\cdot 05 & -78\cdot 3 \\ 31 & 62 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{bmatrix}$$

4.2. *Example* 2

Consider the following transfer-function matrix $[T(s)]$ which is expressed by the product of two relatively prime polynomial matrices $[N(s)]$ and $[D(s)]^{-1}$ or

$$[T(s)]=[N(s)][D(s)]^{-1} \qquad (16)$$

The characteristic matrix polynomial $[D(s)]$ is

$$\begin{aligned}[D(s)] &= B_5 s^4+B_4 s^3+B_3 s^2+B_2 s+B_1 \\ &= C_{11} s^4+C_{21} s^3+C_{12} s^2+C_{22} s+C_{13}\end{aligned}$$

where

$$C_{11} = B_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_{12} = B_3 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad C_{13} = B_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$C_{21} = B_4 = \begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix}, \quad C_{22} = B_2 = \begin{pmatrix} 3 & 4 \\ 4 & 9 \end{pmatrix}$$

It is interesting to determine the stability of this system. Following eqn. (5 *b*) yields the matrix Routh array as follows :

$$\left.\begin{array}{ll}
 & C_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad C_{12} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \quad C_{13} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\
H_1 = \begin{pmatrix} 1{\cdot}4 & -0{\cdot}6 \\ -0{\cdot}6 & 0{\cdot}4 \end{pmatrix} & \\
 & C_{21} = \begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix} \quad C_{22} = \begin{pmatrix} 3 & 4 \\ 4 & 9 \end{pmatrix} \\
H_2 = \begin{pmatrix} \dfrac{3}{2{\cdot}2} & \dfrac{4}{2{\cdot}2} \\ \dfrac{4}{2{\cdot}2} & \dfrac{9}{2{\cdot}2} \end{pmatrix} & \\
 & C_{31} = \begin{pmatrix} 1{\cdot}2 & -0{\cdot}2 \\ 0{\cdot}2 & 1{\cdot}8 \end{pmatrix} \quad C_{32} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\
H_3 = \begin{pmatrix} 11{\cdot}6 & -5{\cdot}4 \\ -5{\cdot}4 & 4{\cdot}6 \end{pmatrix} & \\
 & C_{41} = \begin{pmatrix} \frac{3}{11} & \frac{4}{11} \\ \frac{4}{11} & \frac{9}{11} \end{pmatrix} \\
H_4 = \begin{pmatrix} \frac{3}{22} & \frac{4}{22} \\ \frac{4}{22} & \frac{9}{22} \end{pmatrix} & \\
 & C_{51} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}
\end{array}\right\} \quad (17)$$

The matrix quotients H_i, $i = 1, \ldots, 4$, in the matrix Routh array are positive-definite symmetric real matrices. Therefore the system is stable. It is observed that the block element C_{31} in the first column of the matrix Routh array is not symmetric.

4.3. *Example* 3

Consider the stability and the structure of the matrix Routh array of the following matrix polynomial $T(s)$ are of interest :

$$\begin{aligned} T(s) &= B_4 s^3 + B_3 s^2 + B_2 s + B_1 \\ &= C_{11}' s^3 + C_{21}' s^2 + C_{12}' s + C_{22}' = 0 \end{aligned} \tag{18 a}$$

where

$$C_{11}' = B_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_{12}' = B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C_{21}' = B_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad C_{22}' = B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The determinant of $B_4(=C_{11}') = -1$ and that of $B_1(=C_{22}') = 1$. From the derived necessary conditions for asymptotic stability we conclude that the system is unstable because the determinants of B_4 and B_1 have different sign. Further study of the stability is not necessary. It might be interesting to see the corresponding characteristic equation of this system which can be written as follows :

$$\det T(s) = -s^6 - 2s^5 + 3s^2 + 2s + 1 = 0 \tag{18 b}$$

Because the first and the last coefficients, which are the determinants of B_4 and B_1 respectively, have different sign, therefore the system is unstable. In order to study the structure of the matrix Routh array of this unstable system and the numerically ill-conditioned problem (i.e. $C_{i,1}$ is singular) we apply the matrix Routh algorithm in eqn. (5) and use the $C_{i,j}'$ in eqn. (18 *a*). The matrix Routh array cannot be obtained because C_{21}' is singular. This is a numerically ill-conditioned case. Since the stability is invariant between the original system $T(s)$ and the modified system $T_1(s)(=T(s)|_{s\to 1/s})$, we can construct the matrix Routh array for the $T_1(s)$. The $T_1(s)$ can be written as follows :

$$\begin{aligned} T_1(s) = T(s)|_{s\to 1/s} &= C_{22}' s^3 + C_{12}' s^2 + C_{21}' s + C_{11}' \\ &= C_{11} s^3 + C_{21} s^2 + C_{12} s + C_{22} = 0 \end{aligned} \tag{18 c}$$

where

$$C_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_{12} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$C_{21} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_{22} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The corresponding matrix Routh array is

$$
\left.
\begin{array}{l}
C_{11}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad C_{12}=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\\[6pt]
H_1=C_{11}C_{21}^{-1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\[6pt]
C_{21}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad C_{22}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\[6pt]
H_2=C_{21}C_{31}^{-1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\[6pt]
C_{31}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\[6pt]
H_3=C_{31}C_{41}^{-1}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\[6pt]
C_{41}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}
\end{array}
\right\} \qquad (18\,d)
$$

Although the C_{12} is singular, we can determine all the H_i's. It is observed that the H_1 and H_2 are symmetric and positive definite matrices, while the H_3 is a symmetric and non-positive definite matrix.

An alternative method can be described as follows. Let us construct a new matrix polynomial $T_2(s)$ by multiplying a matrix polynomial $E(s)=(s+1)I$ to the $T(s)$ and then defining the matrix coefficients as $C_{1,i}'$ and $C_{2,i}'$:

$$T_2(s)=(s+1)T(s)=C_{11}'\,s^4+C_{21}'\,s^3+C_{12}'\,s^2+C_{22}'\,s+C_{13}'=0 \qquad (18\,e)$$

where

$$C_{11}'=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad C_{12}'=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}, \quad C_{13}'=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$C_{21}'=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}, \quad C_{22}'=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$$

If we wish to maintain the consistency of $C_{11}=I$, we may interchange the rows in the $T_2(s)$ and define new matrix coefficients as $C_{1,i}$ and $C_{2,i}$:

$$T_2'(s)=C_{11}s^4+C_{21}s^3+C_{12}s^2+C_{22}s+C_{13}=0 \qquad (18\,f)$$

where

$$C_{11}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad C_{12}=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix},\quad C_{13}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$C_{21}=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix},\quad C_{22}=\begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix}$$

The corresponding matrix Routh array is

$$\left.\begin{array}{ll}
 & C_{11}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad C_{12}=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix} \quad C_{13}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\
H_1=\frac{1}{3}\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix} & \\
 & C_{21}=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} \quad C_{22}=\begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix}\\
H_2=\frac{1}{7}\begin{pmatrix}8 & 1\\ 1 & 8\end{pmatrix} & \\
 & C_{31}=\frac{1}{3}\begin{pmatrix}5 & 2\\ 2 & 5\end{pmatrix} \quad C_{32}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\
H_3=\frac{1}{15}\begin{pmatrix}17 & 32\\ 32 & 17\end{pmatrix} & \\
 & C_{41}=\frac{1}{7}\begin{pmatrix}-1 & 6\\ 6 & -1\end{pmatrix}\\
H_4=\frac{1}{7}\begin{pmatrix}6 & -1\\ -1 & 6\end{pmatrix} & \\
 & C_{51}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}
\end{array}\right\} \quad (18\,g)$$

No singular matrix appears in the matrix Routh array and all the H_i's can be obtained. It is observed that only the H_3 is a symmetric but non-positive definite matrix.

From the above illustrations we conclude that if any ill-conditioned problem occurs in the calculation, then the above methods can be applied to solve the problem.

5. Conclusion

The transformation matrix established by Chen and Chu (1966) for transforming the companion form to the Schwarz form has been modified and extended to transform the companion block form to the block Schwarz form. The new matrix in the block Schwarz form has been constructed by using the matrix quotients obtained from the matrix Routh array which is constructed from the characteristic matrix polynomial. When the matrix quotients in the matrix Routh array are positive-definite symmetric real matrices, the sufficient condition derived in this paper shows that the multivariable system is asymptotically stable. Also, a set of necessary conditions has been derived for the asymptotic stability. Thus, we have partially extended the Routh criterion (Routh 1877) to the matrix Routh criterion for a class of matrix polynomials. The direct extension of the necessary and sufficient condition of the Routh criterion (Routh 1877) to a general matrix polynomials need further studies.

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TRANSFER FUNCTION FITTING FROM EXPERIMENTAL FREQUENCY-RESPONSE DATA

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Abstract—A simple method is proposed that will fit the coefficients of a transfer function from the real and imaginary parts of experimental frequency-response data. An approximate logarithmic amplitude-frequency plot is used to formulate an irrational transfer function which then estimates the interpolation data and the degree of the final transfer function. The present method is applicable to either minimum or non-minimum phase system identification.

1. INTRODUCTION

The problem of finding unknown coefficients of a transfer function as a ratio of two frequency-dependent polynomials has been investigated by Levy[1], Kardashov and Karniushin[2], and Sanathanan and Koerner[3]. In general, they would evaluate the polynomial coefficients by minimizing the weighted sum of squares of the errors in magnitude at arbitrary experimental points. Ausman[4] proposed a graphical method to rapidly estimate the coefficients of a transfer function; however, that procedure is only applicable for a minimum phase system.

In this paper a simple method is presented to approximate the coefficients of a transfer function for minimum and non-minimum phase systems. The generalized Bode plot is used to formulate an irrational transfer function from which we obtain interpolation frequency-response data that will allow us to estimate the polynomial coefficients without minimizing the weighted sum of squares of the errors in magnitude at arbitrary points.

2. THE DERIVATION

Consider the transfer function

$$G(s) = \frac{p_0 + p_1 s + p_2 s^2 + \cdots + p_m s^m}{1 + q_1 s + q_2 s^2 + \cdots + q_n s^n} \tag{1}$$

where p_i and q_i are unknown coefficients to be determined. Substituting $s = j\omega_k$ we have

$$\begin{aligned} G(j\omega_k) &= \frac{(p_0 - p_2\omega_k^2 + p_4\omega_k^4 - p_6\omega_k^6 + \cdots) + j(p_1\omega_k - p_3\omega_k^3 + p_5\omega_k^5 - p_7\omega_k^7 + \cdots)}{(1 - q_2\omega_k^2 + q_4\omega_k^4 - q_6\omega_k^6 + \cdots) + j(q_1\omega_k - q_3\omega_k^3 + q_5\omega_k^5 - q_7\omega_k^7 + \cdots)} \\ &= R(\omega_k) + jI(\omega_k) \\ &= R_k + jI_k \end{aligned} \tag{2}$$

where R_k and I_k are the real and imaginary parts of the transfer function at the experimental frequencies ω_k. After we multiply both sides of eqn (2) by the common denominator, we separate the real and imaginary parts and then equate the respective real and imaginary parts. We now have

$$p_0 - p_2\omega_k^2 + p_4\omega_k^4 - p_6\omega_k^6 + \cdots + q_1 I_k\omega_k + q_2 R_k\omega_k^2 - q_3 I_k\omega_k^3 - q_4 R_k\omega_k^4 + \cdots = R_k \tag{3}$$

and

$$p_1\omega_k - p_3\omega_k^3 + p_5\omega_k^5 + \cdots - q_1 R_k\omega_k + q_2 I_k\omega_k^2 + q_3 R_k\omega_k^3 - q_4 I_k\omega_k^4 + \cdots = I_k. \tag{4}$$

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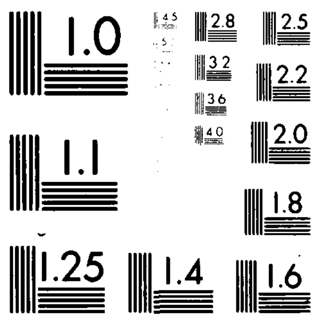

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Combining eqns (3) and (4) results in

$$p_0 + p_1\omega_k - p_2\omega_k^2 - p_3\omega_k^3 + p_4\omega_k^4 + p_5\omega_k^5 - \cdots - q_1(R_k - I_k)\omega_k + q_2(R_k + I_k)\omega_k^2 + q_3(R_k - I_k)\omega_k^3 - q_4(R_k + I_k)\omega_k^4 - \cdots = R_k + I_k \tag{5}$$

The complete form of eqn (5) is

$$\begin{bmatrix} 1 & \omega_1 & -\omega_1^2 & -\omega_1^3 & \omega_1^4 & \omega_1^5 & \ldots & -T_1\omega_1 & s_1\omega_1^2 & T_1\omega_1^3 & -s_1\omega_1^4 & -T_1\omega_1^5 & s_1\omega_1^6 & \ldots \\ 1 & \omega_2 & -\omega_2^2 & -\omega_2^3 & \omega_2^4 & \omega_2^5 & \ldots & -T_2\omega_2 & s_2\omega_2^2 & T_2\omega_2^3 & -s_2\omega_2^4 & -T_2\omega_2^5 & s_2\omega_2^6 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 1 & \omega_m & -\omega_m^2 & -\omega_m^3 & \omega_m^4 & \omega_m^5 & \ldots & -T_m\omega_m & s_m\omega_m^2 & T_m\omega_m^3 & -s_m\omega_m^4 & -T_m\omega_m^5 & s_m\omega_m^6 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 1 & \omega_x & -\omega_x^2 & -\omega_x^3 & \omega_x^4 & \omega_x^5 & \ldots & -T_x\omega_x & s_x\omega_x^2 & T_x\omega_x^3 & -s_x\omega_x^4 & -T_x\omega_x^5 & s_x\omega_x^6 & \ldots \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \cdot \\ p_m \\ q_1 \\ \cdot \\ q_n \end{bmatrix} = \begin{bmatrix} R_1 + I_1 \\ R_2 + I_2 \\ \\ R_m + I_m \\ \cdot \\ \cdot \\ R_x + I_x \end{bmatrix} \tag{6}$$

where

$$s_k = R_k + I_k; \quad k = 1, 2, \ldots$$
$$T_k = R_k - I_k; \quad k = 1, 2, \ldots$$
$$x = m + n + 1$$

By substituting the selected x sets of frequency response data into eqn (6), we can solve for the required unknown coefficients p_i and q_i.

3. ESTIMATION OF THE CORNER FREQUENCY AND THE ORDER

Bode[5] uses piecewise linear segments with integer slopes to approximate the logarithmic amplitude-frequency characteristic of a function. Ausman[4] applies this characteristic to evaluate the coefficients of a transfer function. Polonnikov[6, 7] generalizes Bode's approach to estimate the phase-frequency characteristic. We shall now obtain a logarithmic amplitude-frequency characteristic by piecewise linear segments with accurate integer or fractional slopes. The approximate transfer function is

$$F(s) = \frac{k\left(1+\frac{s}{b_1}\right)^{m_1}\left(1+\frac{s}{b_2}\right)^{m_2}\cdots\left(1+\frac{s}{b_i}\right)^{m_i}}{\left(1+\frac{s}{a_1}\right)^{n_1}\left(1+\frac{s}{a_2}\right)^{n_2}\cdots\left(1+\frac{s}{a_j}\right)^{n_j}} \tag{7}$$

where a_j and b_i are corner frequencies, and where m_i and n_j may be integer or fractional values. In general, eqn (7) is an irrational transfer function. Compared to an approximation made by other methods[4, 5], this present analysis is much better because the slopes may be precise fractional values. However, the worst errors caused by piecewise segment approximation occur at the corner frequencies a_x and b_x; therefore, these corner frequencies provide the most important information of the frequency-response curve. If the interpolation data in eqn (6) include these important corner frequencies, a good transfer-function fitting is expected. In this paper these corner frequency-response data are chosen as main interpolation points for determining the unknown coefficients in eqn (6). The difference of the order of two polynomials in eqn (1) can be estimated from eqn (7). In other words

$$n - m \doteqdot \sum_{x=1}^{j} n_x - \sum_{k=1}^{i} m_k. \tag{8}$$

Based on eqn (8), the numbers of the unknown coefficients and the interpolation points may be estimated.

4. ILLUSTRATIVE EXAMPLES

Example 1. Consider Levy's non-minimum phase example. The frequency-response data generated from the transfer function in eqn (9) is shown in Table 1 and the log-amplitude plot versus log-frequency is shown in Fig. 1.

$$T(s) = \frac{1-s}{1+0.1s+0.01s^2}. \tag{9}$$

The irrational transfer function approximated from the generalized Bode plot is

$$T(s) \doteqdot \frac{\left(1+\frac{s}{0.5}\right)^{0.54}\left(1+\frac{s}{2}\right)^{0.57}\left(1+\frac{s}{40}\right)^{0.15}}{\left(1+\frac{s}{10}\right)^{2.25}} \tag{10}$$

where the corner frequencies are

$$\omega_1 = \omega_3' = 0.5, \qquad \omega_3 = \omega_9' = 10$$
$$\omega_2 = \omega_6' = 2\ , \qquad \omega_4 = \omega_{11}' = 40$$

Table 1.

		Given data				Identified results			
k	ω_k'	$\|T(j\omega_k')\|$	$\angle T(j\omega_k')$	R_k	I_k	$\|G(j\omega_k')\|$	$\angle G(j\omega_k')$	R_k	I_k
1	0.1	1.0064	−6.45	1.0000	−0.1130	1.0013	−6.26	0.9953	−0.1092
2	0.2	1.0239	−12.41	1.0000	−0.2200	1.0160	−12.41	0.9923	−0.2183
3	0.5	1.1194	−29.43	0.975	−0.5500	1.1142	−29.34	0.9713	−0.5459
4	0.7	1.2393	−39.01	0.9630	−0.7800	1.2171	−38.91	0.9472	−0.7644
5	1.0	1.4399	−51.06	0.9050	−1.1200	1.4125	−50.66	0.8955	−1.0924
6	2.0	2.2772	−75.04	0.5880	−2.2000	2.2631	−75.15	0.5798	−2.1875
7	4.0	4.4375	−102.0	−0.925	−4.3400	4.3954	−101.50	−0.877	−4.3071
8	7.0	8.1751	−135.9	−5.870	−5.6900	8.0864	−136.08	−5.826	−5.608
9	10.0	10.05	−174.0	−10.00	−1.1050	9.9115	−174.67	−9.869	−0.9206
10	20.0	5.5541	−233.4	−3.310	4.460	5.4612	−233.4	−3.245	4.3926
11	40.0	2.5451	−253.5	−0.7240	2.4400	2.5363	−253.5	−0.714	2.4338
12	70.0	1.4479	−261.0	−0.2270	1.4300	1.4205	−261.0	−0.225	1.4026
13	100	0.9994	−263.5	−0.1130	0.9930	0.9892	−263.6	−0.109	0.9832

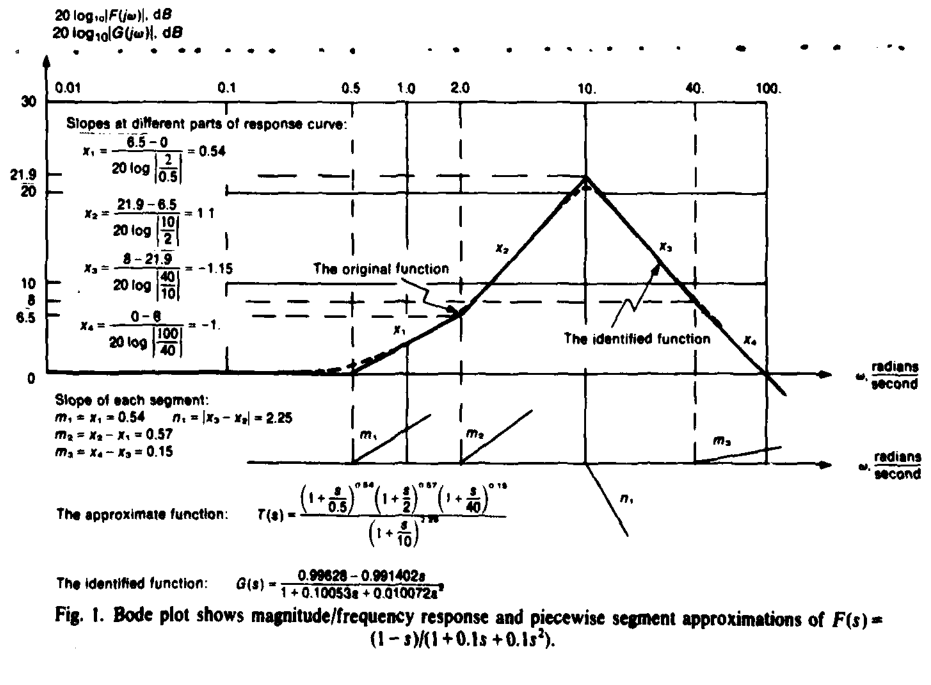

Fig. 1. Bode plot shows magnitude/frequency response and piecewise segment approximations of $F(s) = (1-s)/(1+0.1s+0.1s^2)$.

The order of eqn (1) may be estimated from eqn (10), or

$$m \doteq 0.54+0.57+0.15 \doteq 1$$
$$n \doteq 2.25 \doteq 2$$
$$n-m \doteq 1.$$

Four frequency-response data (ω_1, ω_2, ω_3, ω_4) are required in eqn (6) to fit the four unknown coefficients p_0, p_1, q_1 and q_2. The identified transfer function is

$$G(s)=\frac{0.99628-0.991402s}{1+0.10053s+0.010072s^2}. \tag{11}$$

The corresponding frequency-response data of eqn (11) and that of eqn (9) are compared in Table 1. The results are very satisfactory.

Example 2. A set of frequency-response data generated by the following transfer function is shown in Table 2 and the log-amplitude versus log-frequency plot is shown in Fig. 2.

Table 2.

		Given data				Identified results			
k	ω_k'	$\lvert T(j\omega_k')\rvert$	$\angle T(j\omega_k')$	R_k	I_k	$\lvert G(j\omega_k')\rvert$	$\angle G(j\omega_k')$	R_k	I_k
1	0.1	1.0002	−0.28	1.0002	−0.0048	1.0002	−0.28	1.0002	−0.0048
2	0.4	1.0029	−1.10	1.0027	−0.0193	1.0029	−1.10	1.0028	−0.0193
3	0.8	1.0117	−2.13	1.0110	−0.0375	1.0124	−2.18	1.0116	−0.0385
4	2.0	1.1113	−5.68	1.1058	−0.1101	1.1113	−5.69	1.1058	−0.1101
5	2.2	1.1470	−6.61	1.1394	−0.1321	1.1470	−6.62	1.1394	−0.1322
6	3.6	1.4936	−33.8	1.2418	−0.8299	1.4935	−33.8	1.2416	−0.8300
7	5.4	0.8425	−57.8	0.4485	−0.7132	0.8424	−57.8	0.4484	−0.7132
8	8.0	0.6123	−59.1	0.3147	−0.5253	0.6123	−59.1	0.3147	−0.5252
9	16	0.3730	−69.5	0.1309	−0.3493	0.3730	−69.5	0.1309	−0.3493
10	20	0.3091	−72.9	0.0908	−0.2955	0.3091	−72.9	0.0908	−0.2955
11	100	0.0662	−86.3	0.0042	−0.0661	0.0662	−86.3	0.0042	−0.0661
12	110	0.0602	−86.7	0.0035	−0.0601	0.0602	−86.7	0.0035	−0.0601

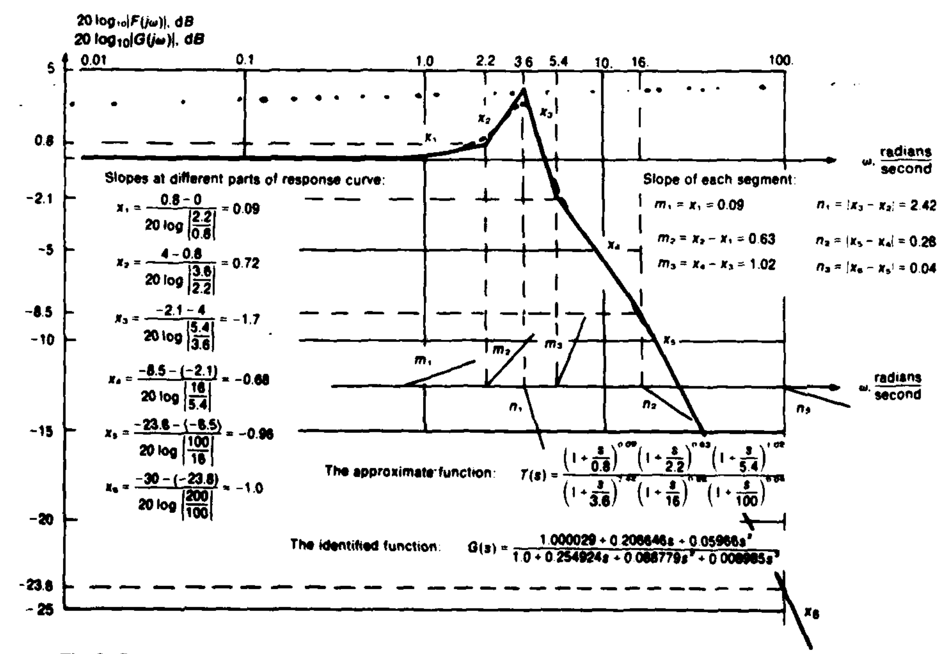

Fig. 2. Bode plot shows magnitude/frequency response and piecewise segment approximations of $F(s)=(6.6378s^2+22.9999s+111.27974)/(s^3+9.8827s^2+28.3706s+111.27974)$.

$$T(s) = \frac{6.6378918s^2 + 22.999878s + 111.27974}{s^3 + 9.882741s^2 + 28.37056s + 111.27974}. \tag{12}$$

The irrational transfer function approximated from the generalized Bode plot is

$$T(s) \doteq \frac{\left(1+\frac{s}{0.8}\right)^{0.09}\left(1+\frac{s}{2.2}\right)^{0.63}\left(1+\frac{s}{5.4}\right)^{1.02}}{\left(1+\frac{s}{3.6}\right)^{2.42}\left(1+\frac{s}{16}\right)^{0.28}\left(1+\frac{s}{100}\right)^{0.04}}. \tag{13}$$

The corner frequencies are

$$\begin{aligned} &\omega_1 = \omega_3' = 0.8, \quad &&\omega_3 = \omega_6' = 3.6, \quad &&\omega_5 = \omega_{10}' = 16 \\ &\omega_2 = \omega_5' = 2.2, \quad &&\omega_4 = \omega_8' = 5.4, \quad &&\omega_6 = \omega_{12}' = 100. \end{aligned} \tag{14}$$

The order of Eqn (1) is estimated as follows:

$$\begin{aligned} m &\doteq 0.09 + 0.63 + 1.02 \doteq 2 \\ n &\doteq 2.42 + 0.28 + 0.04 \doteq 3 \\ n - m &\doteq 1. \end{aligned}$$

At least six unknown coefficients are required to be identified. By substituting the corner frequencies into eqn (6), we have the identified transfer function

$$G(s) = \frac{1.000029 + 0.206648s + 0.05966s^2}{1 + 0.254924s + 0.088779s^2 + 0.008985s^3}. \tag{15}$$

The comparison of the frequency-response data of eqns (12) and (15) is shown in Table 2. These results are also satisfactory.

5. CONCLUSION

A simple method has been presented for fitting a transfer function from experimental frequency-response data. A logarithmic amplitude-frequency curve is first plotted from the available frequency-response data, then it is smoothed and approximated by piecewise segments with integer or fractional slopes. As a result, the most important interpolation data and the order of the transfer function may be obtained from the irrational transfer function. When the slope at two consecutive low frequencies, ω_1 and ω_2, is

$$x(\text{slope}) = \frac{|T(j\omega_1)|_{db} - |T(j\omega_2)|_{db}}{20 \log \left|\frac{\omega_1}{\omega_2}\right|}.$$

(In other words there exists x poles at the origin.), then the available frequency-response data should be multiplied by $(j\omega)^x$ so that eqn (6) may be applied. The method presented in this paper is useful for digital computation and provides an additional tool for system identification.

A computer program, based on the approach discussed, has been written in the appendix.

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APPENDIX

This program is used to fit a transfer function using frequency-response data. The details to prepare the input cards can be summarized as follows:

The first data card:

NDT—number of available frequency-response data

NP—number of different transfer function structures to be identified.

The second data card:

XW_j, $j = 1$ to NDT—a vector of the frequency values at which there is available data

The third data card:

XR_j, $j = 1$ to NDT—a vector of the values of the real parts of the available data at XW_j.

The fourth data card:

XI_j, $j = 1$ to NDT—a vector of the values of the imaginary parts of the available data at XW_j.

The fifth data card:

m—The number of the unknown constants in the numerator polynomial of the transfer function to be identified.

n—The number of the unknown constants in the denominator polynomial of the transfer function to be identified.

The sixth data card:

ND_j, $j = 1$ to NM—A subscript number is assigned to each set of frequency-response data. ND_j is the vector of those subscript numbers which point to the frequency-response data set to be used to identify the transfer function. $NM = n + m$.

The numerical example in Example 1 is used to illustrate the procedure. For the given system, 13 (i.e. $NDT = 13$) frequency-response data are available in Table 1. Various combinations of the structures of the numerator and denominator polynomials may result in various kinds of transfer functions. If we are interested in only one (i.e. $NP = 1$) structure of the transfer function, then

$$T(s) = \frac{p_0 + p_1 s}{1 + q_1 s + q_2 s^2}. \tag{16}$$

The data on the first data card are $NDT = 13$ and $NP = 1$. The values of the frequencies, real parts, and imaginary parts of the available data are given in Table 1. Therefore, the data on the subsequent input cards are

$$\begin{array}{lll} XW_1 = 0.1, & XW_2 = 0.2, \ldots, & XW_{13} = 100 \\ XR_1 = 1.0000, & XR_2 = 1.0000, \ldots, & XR_{13} = -0.1130 \\ XI_1 = -0.1130, & XI_2 = -0.2200, \ldots, & XI_{13} = 0.9930 \end{array}$$

The data on the next card is the number of the unknowns in the numerator and denominator polynomials in eqn (16):

$$m = 2 \quad \text{and} \quad n = 2.$$

The corner frequencies (the most important data) occur at $XW_3 = 0.5$, $XW_6 = 2$, $XW_9 = 10$, and $XW_{11} = 40$; therefore, the values of the selected subscript numbers (i.e., ND_j) are $ND_1 = 3$, $ND_2 = 6$, $ND_3 = 9$, and $ND_4 = 11$. These data appear on the last data card.

The output of this program is $p_0 = 0.99628$, $p_1 = -0.991402$, $q_1 = 0.10053$ and $q_2 = 0.010072$. Also, the real parts, imaginary parts, magnitudes, and phase angles at available frequencies of the identified transfer function in eqn (16) are calculated and printed for comparison with the given data.

A listing of the computer program is as follows:

```
C      A PROGRAM TO FIT TRANSFER FUNCTION USING FREQUENCY-RESPONSE DATA.
C
       DOUBLE PRECISION W(50),XR(50),XI(50),XRI1(50),XRI2(50),B(30),
      1A(30,30),G(30,30),DFTN,H(30),XS,X   ,XW(50)
       DIMENSION ND(30)
       COMPLEX CX,CXX,CXXX,C(50)   ,CXY
 1000  READ (5,501) NDT,NP
 501   FORMAT(16I5)
       WRITE (6,601) NDT,NP
 601   FORMAT((2X,16I5))
       READ(5,502)(XW(J),J=1,NDT)
 502   FORMAT(4F20.8)
       READ(5,502) (XR(J),J=1,NDT)
       READ(5,502) (XI(J),J=1,NDT)
       DO 10 J=1,NDT
 10    WRITE (6,602)  J,XW(J),XR(J),XI(J)
 602   FORMAT (5X,I5,5F20.8)
       DO 90 KM=1,NP
       READ (5,501) M,N
       WRITE (6,601) M,N
```

```
      NM=N+M
      READ (5,501) (ND(J),J=1,NM)
      WRITE (6,601) (ND(J),J=1,NM)
      DO 80 J=1,NM
      JJ=ND(J)
      W(J)=XW(JJ)
      WRITE (6,602) JJ,XW(JJ),XR(JJ),XI(JJ)
      XRI1(J)=XR(JJ)-XI(JJ)
   80 XRI2(J)=XR(JJ)+XI(JJ)
      DO 20 K=1,NM
      B(K)=XRI2(K)
      A(K,1)=1.
      IF (M.EQ.1) GO TO 21
      LK=1
      XS=1.
      DO 30 J=2,M
      A(K,J)=XS*W(K)**(J-1)
      LK=LK+1
      IF(LK.EQ.2) XS=(-1.)*XS
   30 IF(LK.EQ.2) LK=0
   21 CONTINUE
      LK=1
      XS=-1.
      JM=M+1
      DO 40 J=JM,NM
      IF (LK.EQ.1) X=XRI1(K)
      IF (LK.EQ.2) X=XRI2(K)
      IF (LK.EQ.2) XS=(-1.)*XS
      A(K,J)=XS*X*W(K)**(J-JM+1)
      LK=LK+1
   40   IF(LK.GT.2) LK=1
   20  CONTINUE
      CALL INVER (A,NM,G,0,DETN,B,H)
      WRITE (6,603) M,N,(H(J),J=1,NM)
  603 FORMAT (//2X,2I5/(2X,5E20.8)//)
      DO 50 K=1,NDT
      XXS=H(1)
      CX=CMPLX(XXS,0.)
      XX=XW(K)
      IF(M.EQ.1) GO TO 61
      DO 60 J=2,M
      J1=J-1
      CXY=CMPLX(0.,XX)
      CXX=H(J)*CXY**J1
   60 CX=CX+CXX
   61 CXXX=CMPLX(1.,0.)
      DO 70 J=JM,NM
      CXY=CMPLX(0.,XX)
       JJM=J-JM+1
      CXX=H(J)*CXY**JJM
   70 CXXX=CXXX+CXX
      C(K)=CX/CXXX
   50 WRITE (6,604) K,XW(K),C(K)
  604 FORMAT (2X,I5,F20.8,10X,F20.8,5X,F20.8)
      DO 81 J=1,NDT
      XCT=XR(J)
      YCT=XI(J)
      CXY=CMPLX(XCT,YCT)
      XM1=CABS(CXY)
      XT1=ATAN2(YCT,XCT) *57.2958
      CX=C(J)
      XM2=CABS(CX)
      XXCT=REAL(CX)
      YYCT=AIMAG(CX)
      XT2=ATAN2(YYCT,XXCT) *57.2958
      WRITE (6,605) J,XW(J),XR(J),XI(J) ,XM1,XT1
   81 WRITE (6,606) J,XW(J),C(J)  ,XM2,XT2
  605 FORMAT(/2X,I5,F20.8,2X,F20.8,2X,F20.8,2X,E20.8,2X,F20.8)
  606 FORMAT(2X,I5,F20.8,2X,F20.8,2X,F20.8,2X,E20.8,2X,F20.8/)
   90 CONTINUE
      GO TO 1000
      END
      SUBROUTINE INVER (A,N,B,M,DET,XC,XD)
      DOUBLE PRECISION A(30,30),B(30,30),IPVOT(30),INDEX(30,2),
     1PIVOT(30),XC(30),XD(30),DET,T ,S
      EQUIVALENCE (IROW,JROW),(ICOL,JCOL)
  500 FORMAT (I2)
  501  FORMAT (4F20.6)
```

```
601  FORMAT (///(2X,8F15.6))
     M=0
  57 DET=1.
     DO 17 J=1,N
  17 IPVOT(J)=0
     DO 135 I=1,N
     T=0.
     DO 9 J=1,N
     IF(IPVOT(J)-1) 13,9,13
  13 DO 23 K=1,N
     IF(IPVOT(K)-1) 43,23,81
43   IF (DABS(T)-DABS(A(J,K))) 83,23,23
  83 IROW=J
     ICOL=K
     T=A(J,K)
  23 CONTINUE
   9 CONTINUE
     IPVOT(ICOL)=IPVOT(ICOL)+1
     IF(IROW-ICOL) 73,109,73
  73 DET=-DET
     DO 12 L=1,N
     T=A(IROW,L)
     A(IROW,L)=A(ICOL,L)
  12 A(ICOL,L)=T
     IF(M) 109,109,33
  33 DO 2 L=1,M
     T=B(IROW,L)
     B(IROW,L)=B(ICOL,L)
   2 B(ICOL,L)=T
 109 INDEX(I,1)=IROW
     INDEX(I,2)=ICOL
     PIVOT(I)=A(ICOL,ICOL)
     DET=DET*PIVOT(I)
     A(ICOL,ICOL)=1.
     DO 205 L=1,N
 205 A(ICOL,L)=A(ICOL,L)/PIVOT(I)
     IF(M) 347,347,66
  66 DO 52 L=1,M
  52 B(ICOL,L)=B(ICOL,L)/PIVOT(I)
 347 DO 134 LI=1,N
     IF (LI-ICOL) 21,134,21
  21 T=A(LI,ICOL)
     A(LI,ICOL)=0.
     DO 89 L=1,N
  89 A(LI,L)=A(LI,L)-A(ICOL,L)*T
     IF(M) 134,134,18
  18 DO 68 L=1,M
  68 B(LI,L)=B(LI,L)-B(ICOL,L)*T
 134 CONTINUE
 135 CONTINUE
 222 DO 3 I=1,N
     L=N-I+1
     IF(INDEX(L,1)-INDEX(L,2)) 19,3,19
  19 JROW=INDEX(L,1)
     JCOL=INDEX(L,2)
     DO 549 K=1,N
     T=A(K,JROW)
     A(K,JROW)=A(K,JCOL)
     A(K,JCOL)=T
 549 CONTINUE
   3 CONTINUE
     DO 40 K=1,N
40   CONTINUE
     DO 20 K=1,N
     S=0.
     DO 30 J=1,N
30   S=S+A(K,J)*XC(J)
20   XD(K)=S
     WRITE (6,601) (XD(K),K=1,N)
81    CONTINUE
     RETURN
     END
```

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Solution of state-space equations via block-pulse functions

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A recursive algorithm is developed for the piecewise-constant solution of dynamic equations via block-pulse functions $\phi_j(t)$, where $j=1, 2, \ldots, m$. For $1 \leqslant j \leqslant m$ (where j and m are integers) and final time T, each block-pulse function $\phi_j(t)$ is defined by $\phi_j(t)=1$ for $(j-1)T/m \leqslant t < jT/m$ and $\phi_j(t)=0$ otherwise. Compared with Walsh function approaches, the proposed method is simpler to compute, is more suitable for computer programming, and provides the same accuracy. Also, a discrete-time solution is derived for a zero-input state equation.

1. Introduction

Consider a linear time-invariant system described by the state equation

$$\dot{x}(t)=Ax(t)+Bu(t) \tag{1 a}$$

and an initial vector

$$x(0)=x_0 \tag{1 b}$$

where A is an $n \times n$ system matrix, B is an $n \times r$ constant matrix, $x(t)$ is a state vector of n components, $\dot{x}(t)$ is a rate vector, and $u(t)$ is an r-component input vector. It is often difficult to evaluate the integration $\int_0^t \dot{x}(t)\, dt$, which is the solution $x(t)$ in (1), by a numerical method (Carnahan *et al.* 1969). One approach is to find a set of orthogonal functions $\psi_i(t)$ for the approximate solution as follows:

$$x(t)=x(0)+\int_0^t \dot{x}(t)\, dt \cong P\int_0^t \psi(t)\, dt \cong PQ\psi(t)=W\psi(t) \tag{2}$$

where P, Q and W are $n \times m$, $m \times m$ and $n \times m$ weighting matrices, respectively, and $\psi(t)$ is an $m \times 1$ vector with m orthogonal functions $\psi_i(t)$, which are both suitable for approximation of $\dot{x}(t)$ and easy to integrate numerically. Corrington (1973), Chen and Hsiao (1975), and Rao and Sivakumar (1975) chose Walsh functions as the $\psi_i(t)$ for the approximate solution in (2) and reported that their piecewise-constant solution gives a satisfactory result. However, their computational methods (Chen and Hsiao 1975, Rao and Sivakumar 1975) either required the inversion of a large matrix or the inversion of many small matrices. This results in computing time and storage being wasted, and the truncation and round-off errors might be seriously accumulated. Recently, Chen *et al.* (1976) and Gopalsami and Deekshatulu (1976) introduced a set of 'block-pulse functions' for the solutions of distributed systems and identification problems. They pointed out that there is a one-to-one relationship between Walsh functions and block-pulse functions. For $1 \leqslant j \leqslant m$, where j and m are integers, the block-pulse function $\phi_j(t)$ is

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re-defined and extended in the interval $0 \leqslant t < T$ (rather than in the interval $0 \leqslant t < 1$ as in Chen *et al.* (1976), and Gopalsami and Deekshatulu (1976)) and by

$$\phi_j(t) = \begin{cases} 1 & \text{for } (j-1)T/m \leqslant t < jT/m \\ 0 & \text{otherwise} \end{cases} \tag{3 a}$$

T is the final time, and m is the number of subintervals between $t=0$ and $t=T$ as well as the number of block-pulse functions to be used. When m block-pulse functions are used to approximate the integration of the original block-pulse functions, we have

$$\int_0^t \phi(t)\,dt \cong \frac{T}{m} H\phi(t) = \frac{T}{m} \begin{bmatrix} \frac{1}{2} & 1 & 1 & \cdot & 1 \\ 0 & \frac{1}{2} & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \cdot \\ \phi_m(t) \end{bmatrix} \tag{3 b}$$

where $\phi(t)$ is an $m \times 1$ vector with m block-pulse functions. The constant matrix $(T/m)H$, with the dimensions $m \times m$, is the operational matrix (Chen *et al.* 1976, Gopalsami and Deekshatulu 1976) for the block-pulse functions. Sannuti (1976) discussed the properties of the $\phi_j(t)$ and proposed a method for the solutions of linear and non-linear problems. From (3 *b*) we observe that the matrix H is an upper triangular matrix that consists of diagonal elements being $\frac{1}{2}$ and the other elements being 1. By taking advantage of this peculiar arrangement of H and by choosing the block-pulse functions $\phi_j(t)$ as the $\psi(t)$ in (2), an alternative method is proposed in this paper to derive an effective algorithm for the piecewise-constant solution of the state equation in (1). The computation in our algorithm involves the inversion of only one matrix that has the same size as the system matrix. Compared with Walsh function approaches (Corrington 1973, Chen and Hsiao 1975, Rao and Sivakumar 1975) the proposed method is simpler to compute, is more suitable for computer programming, and provides the same accuracy.

2. Main result

Let $x_i(t)$ be the ith component of the state vector $x(t)$ that is the solution of the state equation in (1). The $x_i(t)$ can be expressed approximately as $\sum_{j=1}^{m} C_{i,j}\phi_j(t)$, where m is a large finite number, $\phi_j(t)$ are block-pulse functions, and $C_{i,j}$ are weighting constants to be determined. The state vector $x(t)$ can also be approximated as

$$x(t) \cong C\phi(t) \tag{4 a}$$

where

$$C = [C_1, C_2, \ldots, C_m] \tag{4 b}$$

and

$$\phi(t) = [\phi_1(t), \phi_2(t), \ldots, \phi_m(t)]' \tag{4 c}$$

The prime designates the transpose, and the $n \times m$ matrix C consists of m column vectors C_j to be determined. Our goal is to develop an effective algorithm to determine C_j for every j so that the piecewise-constant solution in (4 a) can be obtained.

We will now derive the recursive algorithm. Let the rate vector $\dot{x}(t)$ in (1) be approximated as

$$\dot{x}(t) \cong D\phi(t) \tag{5 a}$$

by using m block-pulse functions, where

$$D = [d_1, d_2, \ldots, d_m] \tag{5 b}$$

The D is an $n \times m$ constant matrix with m column vectors d_j of size $n \times 1$ to be determined. Integrating (5 a) and using the results of (3 b) and (4 a) yields

$$x(t) \cong D \int_0^t \phi(t)\, dt + x(0) \cong \left[\frac{T}{m} DH + G\right] \phi(t) = C\phi(t) \tag{6 a}$$

where

$$G = [x(0), x(0), \ldots, x(0)] = [g_1, g_2, \ldots, g_m] \tag{6 b}$$

and

$$C = \frac{T}{m} DH + G = [C_1, C_2, \ldots, C_m] \tag{6 c}$$

The g_i in (6 b) is the initial vector $x(0)$, and the constant matrix $(T/m)H$ is shown in (3 b). The accuracy of an approximate solution in (6 a) depends on the number of block-pulse functions and the time interval T/m used. The $r \times 1$ input vector $u(t)$ in (1) can also be approximated as

$$u(t) \cong L\phi(t) \tag{7 a}$$

using m block-pulse functions, where

$$L = [L_1, L_2, \ldots, L_m] \tag{7 b}$$

The $r \times m$ matrix L consists of m column vectors L_j to be determined. By applying the orthogonal property of the block-pulse functions to (7 a), we have

$$L_j \cong \frac{m}{T} \int_{(j-1)T/m}^{jT/m} u(t)\, dt \cong \tfrac{1}{2}[u(jT/m) + u((j-1)T/m)] \tag{7 c}$$

equals average value of $u(t)$ over the interval $(j-1)T/m \leqslant t \leqslant jT/m$. The accuracy of the approximation in (7 c) depends on the time interval T/m used. Substituting (5 a), (6 a) and (7 a) into (1 a) yields

$$D = \frac{T}{m} ADH + AG + BL = \frac{T}{m} ADH + K \tag{8 a}$$

where

$$K = AG + BL = [k_1, k_2, \ldots, k_m] \tag{8 b}$$

The column vector k_j is an $n \times 1$ known vector. The unknown matrix D in (8 a) and (5 a) can be determined from the matrix equation (Chen and Hsiao 1975)

$$\left[I_{nm} - A \otimes \frac{T}{m} H'\right] \begin{bmatrix} d_1 \\ d_2 \\ \cdot \\ d_m \end{bmatrix} = \frac{T}{m} \left[\frac{m}{T} I_{nm} - A \otimes H'\right] \begin{bmatrix} d_1 \\ d_2 \\ \cdot \\ d_m \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \cdot \\ k_m \end{bmatrix} \tag{8 c}$$

or

$$\begin{bmatrix} A_1 & 0 & 0 & \dots & 0 \\ -A & A_1 & 0 & \dots & 0 \\ -A & -A & A_1 & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ -A & -A & -A & \dots & A_1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \cdot \\ d_m \end{bmatrix} = \frac{m}{T} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ \cdot \\ k_m \end{bmatrix} \tag{8 d}$$

where

$$A_1 = \frac{m}{T} I_n - \tfrac{1}{2} A \tag{8 e}$$

The I_{nm} in (8 c) is an $nm \times nm$ identity matrix, and the $\otimes$ in (8 c) represents the Kronecker product. Each $n \times n$ block element 0 in (8 d) is a null matrix and I_n in (8 e) is an $n \times n$ identity matrix. It is known that, as more orthogonal functions are used to approximate $x(t)$, a better approximate solution is obtained. Therefore, m should be a large number and the matrix

$$[(m/T)I_{nm} - A \otimes H']$$

is large. The direct inversion of such a matrix for the solution of d_j in (8 c) is not an effective method as far as the computing time and storage are concerned. However, from the peculiar formulation of the square matrix in (8 d), we can derive an effective algorithm for solving d_j instead of inverting the matrix directly. This effective algorithm is derived in the following way. By pre-multiplying each block element on both sides of (8 d) by A_1^{-1} and by rearranging the new matrix equation, we have an alternative form of (8 d) as

$$\begin{bmatrix} d_2 \\ d_3 \\ d_4 \\ \vdots \\ d_m \end{bmatrix} = \begin{bmatrix} R_2 & 0 & 0 & \dots & 0 \\ R_2 & R_2 & 0 & \dots & 0 \\ R_2 & R_2 & R_2 & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ R_2 & R_2 & R_2 & \dots & R_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \cdot \\ d_{m-1} \end{bmatrix} + \frac{m}{T} \begin{bmatrix} R_1 k_2 \\ R_1 k_3 \\ R_1 k_4 \\ \cdot \\ R_1 k_m \end{bmatrix} \tag{9 a}$$

where

$$\left. \begin{aligned} R_1 &= A_1^{-1} = \left(\frac{m}{T} I_n - \tfrac{1}{2} A \right)^{-1} \\ R_2 &= A_1^{-1} A = R_1 A \\ d_1 &= \frac{m}{T} R_1 k_1 \end{aligned} \right\} \tag{9 b}$$

Equation (9 a) can be solved readily for d_j. After obtaining the matrices R_1 and R_2 and the vector d_1 in (9 b), we can immediately determine the vector d_2 from the first equation in (9 a). Then we can substitute d_2 into the second equation and solve for d_3, etc. Note that the m can be chosen so that $((m/T)I_n - (\frac{1}{2})A)^{-1}$ exists.

The general algorithm is

$$\left.\begin{aligned} d_1 &= \frac{m}{T} R_1 k_1 \\ d_j &= R_2 \sum_{i=1}^{j-1} d_i + \frac{m}{T} R_1 k_j = d_{j-1} + R_2 d_{j-1} + \frac{m}{T} R_1 (k_j - k_{j-1}) \\ & \qquad\qquad\qquad\qquad \text{for } j = 2, 3, \ldots, m \end{aligned}\right\} \quad (10\,a)$$

where

$$\left.\begin{aligned} R_1 &= \left(\frac{m}{T} I_n - \tfrac{1}{2} A\right)^{-1} = A_1^{-1} \\ R_2 &= A_1^{-1} A = R_1 A \end{aligned}\right\} \quad (10\,b)$$

Consequently from (6 c) and (10) we have the required column vectors C_j, or

$$\left.\begin{aligned} C_1 &= \frac{T}{2m} d_1 + g_1 \\ C_j &= \frac{T}{m} \sum_{i=1}^{j-1} d_i + \frac{T}{2m} d_j + g_j = C_{j-1} + \frac{T}{2m} (d_{j-1} + d_j) \quad \text{for } j = 2, 3, \ldots, m \end{aligned}\right\} \quad (11)$$

Substituting (11) into (4) yields the required piecewise-constant solution of the state equation in (1). Note that the $\phi_j(t)$ differs from zero only in the interval $(j-1)T/m \leqslant t < jT/m$; therefore, the jth column vector C_j is the required piecewise-constant solution in that interval. Another advantage of the proposed method is that C_j involves only the vectors d_i, k_i and g_i, for $i = 1 \ldots j$, whereas the Walsh-function approaches (Corrington 1973, Chen and Hsiao 1975, Rao and Sivakumar 1975) require a whole matrix W and a whole vector $\psi(t)$ in (2).

If $u(t) = 0$ in (1), (10) and (11) can be expressed by a set of difference equations

$$d(1) = \frac{m}{T} R_2 x(0) \quad (12\,a)$$

$$d(j+1) = (I_n + R_2) d(j) \quad \text{for } j = 1, 2, \ldots, m-1 \quad (12\,b)$$

and

$$c(1) = \tfrac{1}{2}(2I_n + R_2) x(0) \quad (13\,a)$$

$$c(j+1) = c(j) + \frac{T}{2m} (2I_n + R_2) d(j) \quad \text{for } j = 1, 2, \ldots, m-1 \quad (13\,b)$$

The solution of (12) is

$$d(j) = (I_n + R_2)^{j-1} d(1) = \frac{m}{T} (I_n + R_2)^{j-1} R_2 x(0) \quad (14)$$

Substituting (14) into (13 b) yields

$$c(1) = \tfrac{1}{2}(2I_n + R_2) x(0) \quad (15\,a)$$

$$c(j+1) = c(j) + \tfrac{1}{2}(2I_n + R_2)(I_n + R_2)^{j-1} R_2 x(0) \quad (15\,b)$$

The solution of (15) is

$$c(j+1)=c(1)+\tfrac{1}{2}(2I_n+R_2)\sum_{i=0}^{j-1}(I_n+R_2)^iR_2x(0)$$
$$=\tfrac{1}{2}(2I_n+R_2)\left[I_n+\sum_{i=0}^{j-1}(I_n+R_2)^iR_2\right]x(0)\quad\text{for } j=1,2,\dots \tag{16}$$

Since the trapezoidal rule (as shown in (7)) is used as a base for the numerical integration, or

$$c(j+1)=\frac{x^*(j+1)+x^*(j)}{2} \tag{17 a}$$

where $x^*(j)$ is the discrete-time solution, therefore

$$x^*(j+1)=-x^*(j)+2c(j+1) \tag{17 b}$$

Substituting (16) into (17 b) we have the required discrete-time equations

$$\left.\begin{aligned} x^*(0)&=x(0)\\ x^*(1)&=(I_n+R_2)x(0)\\ x^*(j+1)&=-x^*(j)+(2I_n+R_2)\left[I_n+\sum_{i=0}^{j-1}(I_n+R_2)^iR_2\right]x(0)\end{aligned}\right\} \tag{18 a}$$

The solution of (18 a) is

$$x^*(j)=(I_n+R_2)^jx(0)\quad\text{for } j=0,1,2,\dots \tag{18 b}$$

where $R_2=\left(\frac{m}{T}I_n-\frac{1}{2}A\right)^{-1}A$, $T=$the final time, and the sampling period $=\frac{T}{m}$. Equation (18 b) can be further analysed as

$$x^*(j)=[I_n+R_2]^jx(0)=\Phi^*(j)x(0)\quad\text{for } j=0,1,2,\dots \tag{19 a}$$

where

$$\begin{aligned}\Phi^*(j)&=\text{the transition matrix of a discrete-time system}\\
&=[I_n+R_2]^j\\
&=[I_n+(I_n-\tfrac{1}{2}A\Delta T)^{-1}A\Delta T]^j\quad\text{for } j=0,1,2,\dots,\text{ and } \Delta T=\frac{T}{m}\\
&=[(I_n-\tfrac{1}{2}A\Delta T)^{-1}(I_n+\tfrac{1}{2}A\Delta T)]^j\\
&=[I_n+A\Delta T+\tfrac{1}{2}(A\Delta T)^2+\frac{1}{2^2}(A\Delta T)^3+\frac{1}{2^3}(A\Delta T)^4+\dots]^j\\
&=\left[I_n+A\Delta T+\tfrac{1}{2}(A\Delta T)^2+\sum_{i=3}^{\infty}\frac{1}{2^{i-1}}(A\Delta T)^i\right]^j\end{aligned} \tag{19 b}$$

The exact solution of (1) (with $u(t)=0$) is

$$x(t)=\exp(At)x(0)=\Phi(t)x(0) \tag{20 a}$$

where

$$\begin{aligned}\Phi(t) &= \exp(At) = \text{the transition matrix of a continuous-time system}\\ &= [\exp(A\Delta T)]^j \quad \text{for } j=0,1,2,3,\ldots, \text{ and } t=j\Delta T\\ &= \left[I_n + A\Delta T + \frac{1}{2!}(A\Delta T)^2 + \frac{1}{3!}(A\Delta T)^3 + \frac{1}{4!}(A\Delta T)^4 + \ldots\right]^j\\ &= \left[I_n + A\Delta T + \tfrac{1}{2}(A\Delta T)^2 + \sum_{i=3}^{\infty}\frac{1}{i!}(A\Delta T)^i\right]^j \end{aligned} \tag{20 b}$$

Comparing $\Phi^*(j)$ with $\Phi(t)$ we observe that the first three terms of (19 *b*) are equal to those of (20 *b*), while other terms differ in weighting factors $1/2^{i-1}$ in (19 *b*) and $1/i! = 1/i(i-1)(i-2)\ldots 1$ in (20 *b*). Therefore $\Phi^*(j)$ is a good approximation of $\Phi(t)$ if ΔT is small. Also, we observe that $\Phi^*(j)$ is a finite matrix, while $\Phi(t)$ is an infinite series of matrices, therefore it is more convenient to evaluate $\Phi^*(j)$ than $\Phi(t)$.

It is believed that the derivation of the approximation of $\Phi(t)$ in (20 *b*) by $\Phi^*(j)$ in (19 *b*) is new. When $u(t) \neq 0$, the approximate discrete-time solution $x^*(t)$ of $x(t)$ in (1) can be obtained from (11) and (17 *b*).

3. An illustrative example

Consider the dynamic equation

$$\left.\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ x(0) &= x_0\end{aligned}\right\} \tag{21}$$

where

$$A = \begin{bmatrix}1 & 2\\ 3 & -4\end{bmatrix}, \quad B = \begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}, \quad x(0) = \begin{bmatrix}1\\ 1\end{bmatrix}$$

and

$$u(t) = \text{unit-step functions}$$

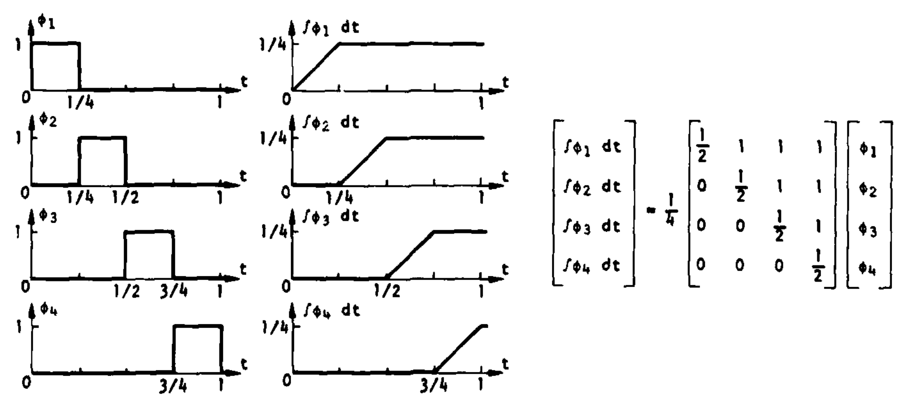

Figure 1. The block-pulse functions and their integrations.

The piecewise-constant solution of the state equation is

$$x(t) \cong C\phi(t) \tag{22}$$

The block-pulse functions $\phi_j(t)$ and the integration of the $\phi_j(t)$ are shown in Fig. 1. The C is an unknown matrix to be determined. The steps to determine C can be listed as follows :

Step 1

Choose $T=1$ s and $m=4$. This means that four block-pulse functions $\phi_j(t)$, $j=1, \ldots, 4$, are used in the interval $0 \leqslant t \leqslant 1$, and the sampling period $= T/m = 0{\cdot}25$ s.

Step 2

Construct G in (6 *b*) and L in (7).

$$G=[x(0), x(0), x(0), x(0)]=[g_1, g_2, g_3, g_4]=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

and

$$L=[L_1, L_2, L_3, L_4]=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Step 3

Calculate K in (8 *b*).

$$K=AG+BL=[k_1, k_2, k_3, k_4]=\begin{bmatrix} 5 & 5 & 5 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Step 4

Determine D in (10).

$$D=[d_1, d_2, d_3, d_4]$$

where

$$R_1=\left(\frac{m}{T} I_2-\tfrac{1}{2}A\right)^{-1}=\begin{bmatrix} 0{\cdot}3077 & 0{\cdot}0510 \\ 0{\cdot}0769 & 0{\cdot}1795 \end{bmatrix}$$

$$R_2=R_1A=\begin{bmatrix} 0{\cdot}4616 & 0{\cdot}4102 \\ 0{\cdot}6154 & -0{\cdot}5641 \end{bmatrix}$$

$$d_1=\frac{m}{T} R_1k_1=\begin{bmatrix} 6{\cdot}3592 \\ 2{\cdot}2560 \end{bmatrix}$$

$$d_2=d_1+R_2d_1+\frac{m}{T} R_1(k_2-k_1)=\begin{bmatrix} 10{\cdot}2200 \\ 4{\cdot}8966 \end{bmatrix}$$

$$d_3=d_2+R_2d_2+\frac{m}{T} R_1(k_3-k_2)=\begin{bmatrix} 16{\cdot}9468 \\ 8{\cdot}4233 \end{bmatrix}$$

$$d_4=d_3+R_2d_3+\frac{m}{T} R_1(k_4-k_3)=\begin{bmatrix} 28{\cdot}2240 \\ 14{\cdot}0999 \end{bmatrix}$$

Step 5

Evaluate the required C in (11).

$$C = [C_1, C_2, C_3, C_4]$$

where

$$C_1 = \frac{T}{2m} d_1 + g_1 = \begin{bmatrix} 1\cdot 7949 \\ 1\cdot 2820 \end{bmatrix}$$

$$C_2 = C_1 + \frac{T}{2m}(d_2 + d_1) = \begin{bmatrix} 3\cdot 8773 \\ 2\cdot 1792 \end{bmatrix}$$

$$C_3 = C_2 + \frac{T}{2m}(d_3 + d_2) = \begin{bmatrix} 7\cdot 3038 \\ 3\cdot 8547 \end{bmatrix}$$

$$C_4 = C_3 + \frac{T}{2m}(d_4 + d_3) = \begin{bmatrix} 13\cdot 0125 \\ 6\cdot 6936 \end{bmatrix}$$

The required piecewise-constant solution in (21) is

$$x_1(t) \cong 1\cdot 7949\phi_1(t) + 3\cdot 8773\phi_2(t) + 7\cdot 3038\phi_3(t) + 13\cdot 0125\phi_4(t)$$
$$x_2(t) \cong 1\cdot 2820\phi_1(t) + 2\cdot 1792\phi_2(t) + 3\cdot 8547\phi_3(t) + 6\cdot 6936\phi_4(t)$$

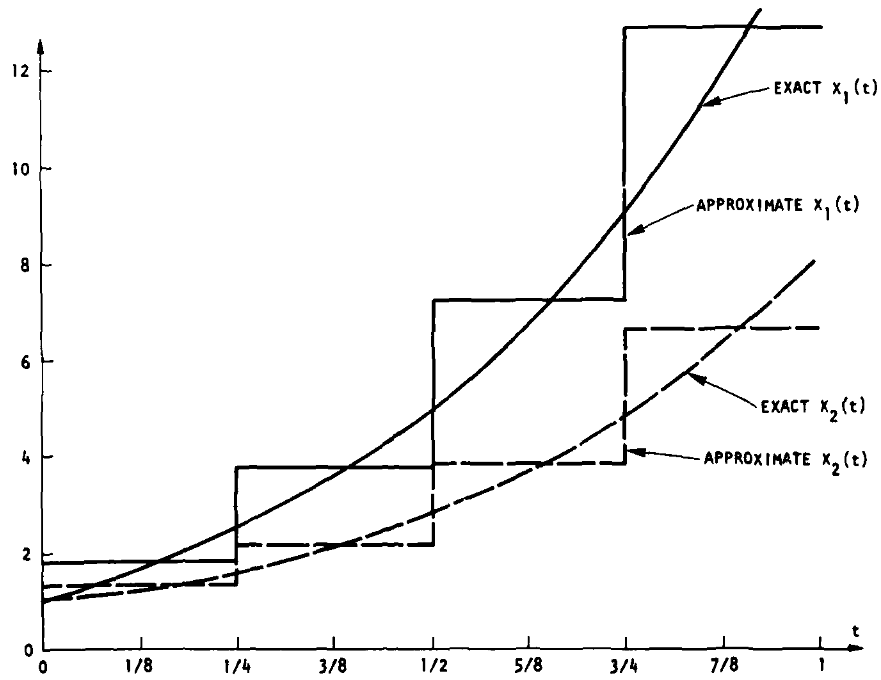

Figure 2. The exact solutions and the approximated solutions.

The exact solution of (21) is

$$x_1(t) = \tfrac{16}{7} \exp(2t) - \tfrac{3}{35} \exp(-5t) - \tfrac{6}{5}$$

$$x_2(t) = \tfrac{8}{7} \exp(2t) + \tfrac{9}{35} \exp(-5t) - \tfrac{2}{5}$$

The response curves of the exact solution and the approximated solution are shown in Fig. 2. The approximate discrete-time solution $x^*(t)$ of $x(t)$ in (21) can be obtained from the C in (22) and (17 *b*).

If $u(t)=0$ in (21), the exact solution of (21) is

$$\left.\begin{aligned} x_1(t) &= \tfrac{8}{7} \exp(2t) - \tfrac{1}{7} \exp(-5t) \\ x_2(t) &= \tfrac{4}{7} \exp(2t) + \tfrac{3}{7} \exp(-5t) \end{aligned}\right\} \tag{23}$$

From (23) and (18) we can evaluate the exact solution $x(t)$ and the approximated solution $x^*(t)$ at samples $j=1, 2, 3, 4$, and sampling period $= T/m = 0{\cdot}25$. The results are tabulated as follows :

j	t	$x_1(t)$	$x_1^*(t)$	$x_2(t)$	$x_2^*(t)$
0	0	1	1	1	1
1	0·25	1·843	1·872	1·065	1·051
2	0·50	3·095	3·167	1·589	1·610
3	0·75	5·119	5·289	2·571	2·651
4	1·00	8·444	8·818	4·225	4·411

It is interesting to observe that the solution obtained by the four-point approximation is quite satisfactory.

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STATE-EQUATION FITTING FROM FREQUENCY-RESPONSE DATA

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Abstract—A method is given for optimally fitting parameter matrices of state equations from the real and imaginary parts of noise free frequency-response data of a multi-input, multi-output, linear dynamic system. It is assumed that all state variables are accessible for measurement. The obtained data may contain measurement errors.

1. INTRODUCTION

Several authors[1–3] have considered the application of frequency response concepts for identification of dynamic systems. The problems of predicting parametric error from frequency response measurements have also been investigated[3–5]. A method is presented here to determine the best estimate, in least mean square sense, of the parameter matrices of the multi-input, multi-output, linear, time-invariant dynamic system equations if all the state variables are accessible for measurement. The obtained data are noise free and contain measurement errors.

2. DERIVATION

The state equations of an asymptotically stable, completely controllable and observable, linear time-invariant system are given by:

$$\dot{\hat{X}} = \hat{A}\hat{X} + \hat{B}\hat{U} \tag{1}$$

$$\hat{Y} = \hat{C}\hat{X} \tag{1a}$$

$$\hat{X}(0) = [0] \tag{1b}$$

where $\hat{A}$ is a constant $n \times n$ system matrix, $\hat{X}$ is an $n \times 1$ state vector, $\hat{B}$ is a constant $n \times r$ input matrix, $\hat{C}$ is a constant $m \times n$ output matrix, $\hat{U}$ is an $r \times 1$ input vector, and $\hat{Y}$ is an $m \times 1$ output vector. Let us define,

$$\hat{B} = [\hat{b}_1, \ldots, \hat{b}_r] \tag{2}$$

$$\hat{C} = \begin{bmatrix} \hat{C}_1{}^T \\ \cdot \\ \cdot \\ \cdot \\ \hat{C}_m{}^T \end{bmatrix} \tag{2a}$$

$$\hat{U} = \begin{bmatrix} U_1 \\ \cdot \\ \cdot \\ \cdot \\ U_r \end{bmatrix} \tag{2b}$$

where $\hat{b}_i$ is an $n \times 1$ column vector and $\hat{C}_i^T$ is an $n \times 1$ row vector.

The Laplace transformation of eqns (1) and (1a) yields,

$$(s\hat{I} - \hat{A})\hat{X}(s) = \hat{B}\hat{U}(s) \tag{3}$$

and

$$\hat{Y}(s) = \hat{C}\hat{X}(s). \tag{3a}$$

Successive choice of each of the scalars $U_e(s)$ in eqn (2b) as an input while the remaining scalar components of $\hat{U}(s)$ are zero yields the following set of transfer functions from each of the scalar inputs to the state variables.

$$(s\hat{I} - \hat{A})\hat{T}_e(s) = \hat{b}_e \tag{4}$$

where

$$\hat{T}_e(s) = \frac{1}{U_e(s)}\hat{X}_e(s) \quad \text{and} \quad e = 1, \ldots, r. \tag{4a}$$

If the input functions of $\hat{U}(s)$ are sinusoidal functions with varying frequencies ω_k, we obtain the corresponding frequency response data $\hat{T}_e(j\omega_k)$ as follows:

$$\hat{T}_e(j\omega_k) = \hat{P}_e(\omega_k) + j\hat{q}_e(\omega_k), \qquad e = 1, \ldots, r \tag{5}$$

where $\hat{P}_e(\omega_k)$ and $\hat{q}_e(\omega_k)$ are vectors of the real and the imaginary parts of $\hat{T}_e(j\omega_k)$.

Multiplying the steady state portion of eqn (4a) by a normalization constant M_e (i.e. the magnitude of a sinusoidal input function) we have

$$\hat{X}_e(j\omega_k) = M_e\hat{T}_e(j\omega_k) = M_e\hat{P}_e(\omega_k) + jM_e\hat{q}_e(\omega_k), \qquad e = 1, \ldots, r \tag{5a}$$

and

$$\hat{X}(j\omega_k) = \sum_{e=1}^{r} \hat{X}_e(j\omega_k) \tag{5b}$$

Substituting $s = j\omega_k$ and eqns (5) and (5a) into eqns (4) and (3a) yields

$$[j\omega_k\hat{I} - \hat{A}][\hat{P}_e(\omega_k) + j\hat{q}_e(\omega_k)] = \hat{b}_e \tag{6}$$

$$\hat{Y}_e(j\omega_k) = \hat{C}[M_e\hat{P}_e(\omega_k) + jM_e\hat{q}_e(\omega_k)] = \hat{g}_e(\omega_k) + j\hat{h}_e(\omega_k) \tag{6a}$$

and

$$\hat{Y}(j\omega_k) = \sum_{e=1}^{r} \hat{Y}_e(j\omega_k) \tag{6b}$$

where $\hat{g}_e(\omega_k)$ and $\hat{h}_e(\omega_k)$ are vectors of the real and imaginary parts of $\hat{Y}_e(j\omega_k)$. After we separate the real and imaginary parts of eqns (6) and (6a) and equate the respective real and imaginary parts, we have

$$\hat{A}\hat{q}_e(\omega_k) = \omega_k\hat{P}_e(\omega_k) \tag{7}$$

$$\hat{A}\hat{P}_e(\omega_k) + \omega_k\hat{q}_e(\omega_k) = -\hat{b}_e \tag{7a}$$

$$\hat{g}_e(\omega_k) = \hat{C}M_e\hat{P}_e(\omega_k) \tag{7b}$$

and

$$\hat{h}_e(\omega_k) = \hat{C}M_e\hat{q}_e(\omega_k). \tag{7c}$$

The parameter matrices $\hat{A}$, $\hat{b}_e$ and $\hat{C}$ can be obtained as follows:

$$\hat{A} = [\omega_1 \hat{P}_e(\omega_1), \omega_2 \hat{P}_e(\omega_2), \ldots, \omega_n \hat{P}_e(\omega_n)][\hat{q}_e(\omega_1), \hat{q}_e(\omega_2), \ldots, \hat{q}_e(\omega_n)]^{-1} \tag{8}$$

$$\hat{b}_e = -[\hat{A}\hat{P}_e(\omega_k) + \omega_k \hat{q}_e(\omega_k)] \tag{8a}$$

and

$$\begin{aligned}\hat{C} &= [\hat{g}_e(\omega_1), \hat{g}_e(\omega_2), \ldots, \hat{g}_e(\omega_n)][M_e\hat{P}_e(\omega_1), M_e\hat{P}_e(\omega_2), \ldots, M_e\hat{P}_e(\omega_n)]^{-1} \\ &= [\hat{h}_e(\omega_1), \hat{h}_e(\omega_2), \ldots, \hat{h}_e(\omega_n)][M_e\hat{q}_e(\omega_1), M_e\hat{q}_e(\omega_2), \ldots, M_e\hat{q}_e(\omega_n)]^{-1} \\ &= \left[\sum_{e=1}^{r} \hat{g}_e(\omega_1), \ldots, \sum_{e=1}^{r} \hat{g}_e(\omega_n)\right]\left[\sum_{e=1}^{r} M_e\hat{P}_e(\omega_1), \ldots, \sum_{e=1}^{r} M_e\hat{P}_e(\omega_n)\right]^{-1} \\ &= \left[\sum_{e=1}^{r} \hat{h}_e(\omega_1), \ldots, \sum_{e=1}^{r} \hat{h}_e(\omega_n)\right]\left[\sum_{e=1}^{r} M_e\hat{q}_e(\omega_1), \ldots, \sum_{e=1}^{r} M_e\hat{q}_e(\omega_n)\right]^{-1}.\end{aligned} \tag{8b}$$

The data in eqns (8)–(8b) can be chosen so that the matrix inversions exist.

3. EVALUATION OF OPTIMAL PARAMETER MATRICES

If the frequency response data are noise free and measurement error free, then there exist unique parameter matrices $\hat{A}$, $\hat{B}$ and $\hat{C}$. However, in practice, there exist measurement errors even if the system is noise free. As a result, the evaluated parameter matrices have inaccuracies due to the errors. In this paper, optimal parameter matrices are evaluated from the measurement error contaminated data.

Consider i sets of parameter matrices $\hat{A}$, $\hat{b}_e$ and $\hat{C}$, which are defined as $\hat{A}_i$, $\hat{b}_{ei}$ and $\hat{C}_i$, and which are evaluated from i sets of data using one control input or r control inputs. If many sets of experimental frequency response data can be obtained, then the optimal parameter matrices $\tilde{A}$, $\tilde{b}_e$ and $\tilde{C}$ can be obtained from the matrix-mean values, or

$$\tilde{A} = \frac{1}{k}\sum_{i=1}^{k} \hat{A}_i \tag{9}$$

$$\tilde{b}_e = \frac{1}{k}\sum_{i=1}^{k} \hat{b}_{ei} \tag{9a}$$

$$\tilde{C} = \frac{1}{k}\sum_{i=1}^{k} \hat{C}_i \tag{9b}$$

However, to obtain many sets of frequency response data is often not practical and sometimes impossible. The following technique is proposed to obtain the optimal matrices with fewer sets of frequency response data. Suppose that the system matrices $\hat{A}_i$, $i = 1, \ldots, r$ can be evaluated by r sets of frequency response data which are obtained from the controllable system by any one input U_e or by r sets of inputs, then we construct the following matrix equation,

$$\hat{E}\tilde{A} = \hat{F} \tag{10}$$

where

$$\hat{E} = \begin{bmatrix} \hat{A}_1^{-1} \\ \hat{A}_2^{-1} \\ \cdot \\ \cdot \\ \cdot \\ \hat{A}_r^{-1} \end{bmatrix} \text{and} \quad \hat{F} = \begin{bmatrix} \hat{I}_1 \\ \hat{I}_2 \\ \cdot \\ \cdot \\ \cdot \\ \hat{I}_r \end{bmatrix} \tag{10a}$$

in which $\hat{A}_i^{-1}$ are $n \times n$ inverse matrices of $\hat{A}_i$ obtained by the use of eqn (8) and $\hat{I}_i$ are $n \times n$ identity matrices. The desired optimal matrix[6, 7] $\tilde{A}$ which minimizes the sum of squares of residuals $\hat{S} = \hat{R}^T\hat{R}$, where $\hat{R} = \hat{F} - \hat{E}\tilde{A}$, is given by

$$\tilde{A} = (\hat{E}^T\hat{E})^{-1}\hat{E}^T\hat{F}. \tag{11}$$

By a similar approach the optimal matrices $\tilde{b}_e$ and $\tilde{C}$ can be obtained as follows:

To obtain $\bar{b}_e$ we construct the matrix equation

$$\hat{G}_e \bar{H}_e = \hat{F} \tag{12}$$

where

$$\hat{G}_e = \begin{bmatrix} \hat{G}_1^{-1} \\ \hat{G}_2^{-1} \\ \cdot \\ \cdot \\ \cdot \\ \hat{G}_r^{-1} \end{bmatrix} \tag{12a}$$

in which $\hat{G}_i^{-1}$ are $n \times n$ inverse matrices of $\hat{G}_i$ and the elements at jth row and kth column in $\hat{G}_i$ and $\bar{H}_e$ are:

$$\begin{aligned} \hat{G}_i(j,k) &= \hat{b}_{ei}(j,1) && \text{if } j = k \\ &= 0 && \text{if } j \neq k \end{aligned} \tag{12b}$$

$$\begin{aligned} \bar{H}_e(j,k) &= \bar{b}_e(j,1) && \text{if } j = k \\ &= 0 && \text{if } j \neq k \end{aligned} \tag{12c}$$

$$j = 1, \ldots, n, \qquad i = 1, \ldots, r$$
$$k = 1, \ldots, n, \qquad e = 1, \ldots, r.$$

It is interesting to note the fact that $\hat{G}_i(j,k)$ and $\bar{H}_e(j,k)$ are diagonal which greatly reduces the practical problem of calculating $\bar{H}_e$.

The optimal matrix $\bar{H}_e$ is

$$\bar{H}_e = (\hat{G}_e^T \hat{G}_e)^{-1} \hat{G}_e^T \hat{F}. \tag{13}$$

The optimal vector $\bar{b}_e$ can be obtained from eqn (12c). To obtain the optimal row vector $\hat{C}_z^T$ in $\hat{C}$ we use the following matrix equation:

$$\hat{D}_z \bar{S}_z = \hat{F} \tag{14}$$

where

$$\hat{D}_z = \begin{bmatrix} \hat{D}_1^{-1} \\ \hat{D}_2^{-1} \\ \cdot \\ \cdot \\ \cdot \\ \hat{D}_r^{-1} \end{bmatrix} \tag{14a}$$

in which $\hat{D}_i^{-1}$ are $n \times n$ inverse matrices of $\hat{D}_i$ and the elements of the jth row and kth column in $\hat{D}_i$ and $\bar{S}$ are

$$\begin{aligned} \hat{D}_i(j,k) &= \hat{C}_{zi}^T(1,j) && \text{if } j = k \\ &= 0 && \text{if } j \neq k \end{aligned} \tag{14b}$$

$$\begin{aligned} \bar{S}_z(j,k) &= \bar{C}_z^T(1,j) && \text{if } j = k \\ &= 0 && \text{if } j \neq k \end{aligned} \tag{14c}$$

$j = 1, \ldots, n$, $i = 1, \ldots, r$, $z = 1, \ldots, m$, $k = 1, \ldots, n$. Here again the structure of $\hat{D}_i(j,k)$ is quite favorable for performing the necessary inversions.

The optimal matrix $\bar{S}_z$ can be obtained from

$$\bar{S}_z = (\hat{D}_z^T \hat{D}_z)^{-1} \hat{D}_z^T \hat{F}. \tag{15}$$

The optimal row vector $\bar{C}_z^T$ can be obtained from eqn (14c). After obtaining the optimal vectors $\bar{b}_e$ and $\bar{C}_z^T$ we have the optimal input matrix and output matrix, or

$$\bar{B} = [\bar{b}_1, \ldots, \bar{b}_r] \tag{16}$$

and

$$\bar{C} = \begin{bmatrix} \bar{C}_1^T \\ \cdot \\ \cdot \\ \cdot \\ \bar{C}_m^T \end{bmatrix} \tag{17}$$

4. CONCLUSION

A method for the solution of the difficult problem of identifying a multi-input, multi-output, linear system from measurement error contaminated data has been presented. The resultant parameter matrices are optimal in the least mean square sense. The particular advantage of this technique is the ability to utilize a relatively limited amount of experimental data to obtain the systems dynamic equations. The identification process can be easily performed using digital computers.

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APPENDIX

Illustrative example. For a known dynamic system described by the following state equation,

$$\begin{aligned} \dot{\hat{X}} &= \hat{A}\hat{X} + \hat{B}\hat{U} \\ \hat{Y} &= \hat{C}\hat{X} \end{aligned} \tag{18}$$

where

$$\hat{A} = \begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{and } \hat{C} = \begin{bmatrix} 1 & 0.5 \\ 0 & 2 \end{bmatrix}$$

the error contaminated frequency response of Table 1 was obtained.

Assuming a unity magnitude for the excitation function or M_1 and M_w equal unity and by following eqns (8), (8a) and (8b), we have

$$\hat{A}_1 = \begin{bmatrix} -1.018087614 & -0.967404614 \\ 1.963824775 & -3.934809232 \end{bmatrix} \tag{19}$$

$$\hat{A}_2 = \begin{bmatrix} -0.993028846 & -1.015416667 \\ 2.005288462 & -4.00249999 \end{bmatrix}$$

Table 1. Frequency response data

ω_k	$e = 1$, $P_1(\omega_k)$	$T_e(j\omega_k)$ $q_1(\omega_k)$	$e = 2$, $P_2(\omega_k)$	$T_e(j\omega_k)$ $q_2(\omega_k)$	$e = 1$, $g_1(\omega_k)$	$Y_e(j\omega_k)$ $h_1(\omega_k)$	$e = 2$, $g_2(\omega_k)$	$Y_e(j\omega_k)$ $h_2(\omega_k)$
0.2	0.658	−0.077	−0.821	0.104	0.82	−0.10	−0.91	0.13
	0.326	−0.055	−0.158	0.060	0.65	−0.11	−0.32	0.12
2.0	0.269	−0.346	−0.288	0.442	0.29	−0.44	−0.20	0.51
	0.038	−0.192	0.173	0.135	0.08	−0.38	0.35	0.27

for $e = 1$, we have

$$\hat{b}_{e1} = \begin{bmatrix} 1.000275632 \\ -0.001001583 \end{bmatrix}, \quad \hat{b}_{e2} = \begin{bmatrix} 1.000208233 \\ 0.0011505081 \end{bmatrix} \tag{19a}$$

and when $e = 2$, we have

$$\hat{b}_{e1} = \begin{bmatrix} -1.003112556 \\ 0.989774387 \end{bmatrix}, \quad \hat{b}_{e2} = \begin{bmatrix} -1.00220058 \\ 0.987559404 \end{bmatrix} \tag{19b}$$

When $z = 1$, we have

$$\hat{C}_{z1}^T = [1.011006541, \quad 0.474716861], \qquad C_{z2}^T = [1.007961095, \quad 0.521923674] \tag{19c}$$

and for $z = 2$, we obtain

$$\hat{C}_{z1}^T = [0.0220130807, \quad 1.949433722], \qquad C_{z2}^T = [0.0003199367, \quad 2.023653999] \tag{19d}$$

Applying eqns (11), (13) and (15) we have the optimal parameter matrices

$$\bar{A} = \begin{bmatrix} -1.005 & -0.991 \\ 1.983 & -3.966 \end{bmatrix} \tag{20}$$

$$\bar{B} = \begin{bmatrix} 1.000 & -1.002 \\ 0.000 & 0.988 \end{bmatrix} \tag{20a}$$

and

$$\bar{C} = \begin{bmatrix} 1.009 & 0.496 \\ 0.000 & 1.985 \end{bmatrix}. \tag{20b}$$

Compared with eqn (18), the answer is quite satisfactory.

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An algebraic method to determine the common divisor, poles and transmission zeros of matrix transfer functions

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A purely algebraic method which uses the matrix Routh algorithm and its reverse process of the algorithm is presented to decompose a matrix transfer function into a pair of right co-prime polynomial matrices or left co-prime polynomial matrices. The poles and transmission zeros of the matrix transfer function are determined from a pair of relatively prime polynomial matrices. Also, the common divisor of two matrix polynomials can be obtained by using the matrix Routh algorithm and the matrix Routh array.

1. Introduction

The properties and applications of poles and transmission zeros of a multivariable system have been extensively studied in recent years by many researchers (Desoer and Schulman 1974, Kwakernaak and Sivan 1972, Rosenbrock 1970, Moore and Silverman 1972, Wolovich 1972, 1973, Davison and Wang 1974, Francis and Wonham 1975, Sinswat *et al.* 1976, Kouvaritakis and MacFarlane 1976, Wang and Desoer 1972). Desoer and Schulman (1974) defined the poles as real or complex numbers for which the responses of a circuit or system to a series of singular inputs are purely exponential. The transmission zeros are also defined as real or complex numbers for which the transmission of some particular signals is completely blocked. The role of poles in the analysis and synthesis of circuits and systems is well known, and in recent years the transmission zeros are found to be important in many aspects of feedback control theory (Desoer and Schulman 1974, Kwakernaak and Sivan 1972, Rosenbrock 1970, Moore and Silverman 1972, Wolovich 1972, 1973, Davison and Wang 1974, Francis and Wonham 1975, Sinswat *et al.* 1976, Kouvaritakis and MacFarlane 1976, Wang and Desoer 1972). Therefore, it is useful and desirable to have an effective method to determine the locations of these poles and transmission zeros. Several methods are available to locate the positions of these poles and zeros (Kwakernaak and Sivan 1972, Rosenbrock 1970, Moore and Silverman 1972, Wolovich 1973, Davison and Wang 1974, Francis and Wonham 1975, Sinswat *et al.* 1976, Kouvaritakis and MacFarlane 1976). However, most of the suggested approaches (Rosenbrock 1970, Moore and Silverman 1972, Wolovich 1973, Davison and Wang 1974, Francis and Wonham 1975, Sinswat *et al.* 1976, Kouvaritakis and MacFarlane 1976) are derived for the systems which are represented by state equations in the time domain. The major disadvantage of most time-domain approaches is that the computation may not be very attractive if the dynamic systems are

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of high order. When a given multivariable system is described by a matrix transfer function that might have a high degree common divisor (the common factor) of the numerator and denominator matrix polynomials, the order of the corresponding state equations is in general very high. Therefore, most time-domain approaches may be difficult to apply. In this paper, a purely algebraic method is derived in the frequency domain for the determination of the poles and transmission zeros of a matrix transfer function. The matrix Routh algorithm and the reverse process of the algorithm (Shieh and Gaudiano 1974, Shieh 1975, Shieh *et al.* 1975) are used to decompose an $n_0 \times n_i$ rational matrix transfer function $T(s)$ into $D_l(s)^{-1}N_l(s)$ and $N_r(s)D_r(s)^{-1}$, where the polynomial matrices $D_l(s)$ and $N_l(s)$ with appropriate size are left co-prime and $N_r(s)$ and $D_r(s)$ right co-prime. When $n_0 = n_i$, the poles (the transmission zeros) of the $T(s)$ are determined from the zeros of the determinant $D_l(s)$ or $D_r(s)$ ($N_l(s)$ or $N_r(s)$). When $n_0 \neq n_i$, or the matrix Routh algorithm is of ill-conditioned case, the determinant of the rectangular polynomial matrices $N_l(s)$ and $N_r(s)$ cannot be obtained. An $n_i \times n_0$ matrix transfer function $T^+(s)$, which is the generalized inverse (Desoer and Schulman 1974) of the modified $T(s)$, is established and factored into $D_l^*(s)^{-1}N_l^*(s)$ when $n_0 \geqslant n_i$ or $N_r^*(s)D_r^*(s)^{-1}$ when $n_0 \leqslant n_i$. The transmission zeros of the $T(s)$ are determined from the invariant poles of the $T^+(s)$ or from the zeros of the determinant $D_l^*(s)$ with size $n_i \times n_i$ or $D_r^*(s)$ having size $n_0 \times n_0$.

Along the same line, recently, several approaches have been proposed by various authors (Kung *et al.* 1976, Anderson and Jury 1976, Emre and Silverman 1976) to determine the relative primeness of two polynomial matrices. The generalized resultant matrix (Barnett 1971) and the generalized Bezoutian and Sylvester matrices are used in their works. When the degree of the polynomial matrices that might have a high degree common divisor is high, the dimension of the resultant matrix or the equivalent test matrix is very high. As a result, the effectiveness of their approaches is less.

2. The matrix Routh algorithm and the matrix Routh array

In a single variable system it is well known that the poles and zeros of a transfer function can be determined from the respective denominator and numerator polynomials that are relatively prime. The Routh algorithm and the Routh array (Fryer 1959) are often used to determine the common factor of the two polynomials in order to determine the pair of relatively prime polynomials. In this paper we extend the concept to a multivariable system that is described by a matrix transfer function. Let us define that R and C denote the field of real numbers and complex numbers, respectively, and $R[s]$ and $R(s)$ the sets of all polynomials and rational functions in the field of complex variables having real coefficients. We also define that $R[s]^{n_0 \times n_i}$ and $R(s)^{n_0 \times n_i}$ are the sets of all $n_0 \times n_i$ matrices with elements in $R[s]$ and $R(s)$, respectively.

Consider the following matrix transfer function $T(s) \in R(s)^{n_0 \times n_i}$ which is a product of a polynomial matrix $A_2(s) \in R[s]^{n_0 \times n_i}$ and the inverse of another polynomial matrix $A_1(s) \in R[s]^{q \times q}$, where $q = \min(n_0, n_i)$:

$$T(s) = A_2(s)A_1(s)^{-1} = [A_{21} + A_{22}s + \ldots + A_{2,n}s^{n-1}] \times [A_{11} + A_{12}s + \ldots + A_{1,n+1}s^n]^{-1} \qquad (1\,a)$$

$$T(s) = A_1(s)^{-1}A_2(s) = [A_{11} + A_{12}s + \ldots + A_{1,n+1}s^n]^{-1} \times [A_{21} + A_{22}s + \ldots + A_{2,n}s^{n-1}] \quad (1\,b)$$

where

$$A_2(s) = \sum_{i=1}^{n} A_{2,i}s^{i-1} \quad \text{and} \quad A_1(s) = \sum_{i=1}^{n+1} A_{1,i}s^{i-1}$$

The matrix coefficients in the $A_2(s)$ and $A_1(s)$ are expressed by the double subscript notation as $A_{2,i} \in R^{n_0 \times n_i}$ and $A_{1,i} \in R^{q \times q}$ for the use of the matrix Routh algorithm. If the $T(s)$ is expressed as follows :

$$T(s) = \frac{1}{\Delta_0(s)} \Phi(s) \quad (2)$$

then

$$\Delta_0(s) = \sum_{i=1}^{n+1} a_i s^{i-1}, \quad A_1(s) = \sum_{i=1}^{n+1} a_i I_q s^{i-1} = \sum_{i=1}^{n+1} A_{1,i}s^{i-1}$$

and

$$\Phi(s) = \sum_{i=1}^{n} \Phi_i s^{i-1} = \sum_{i=1}^{n} A_{2,i}s^{i-1}$$

where $\Delta_0(s) \in R[s]$ is a polynomial and $I_q \in R^{q \times q}$ is an identity matrix. By using the following matrix Routh algorithm and the reverse process of the algorithm, the $T(s)$ can be factored into $D_l(s)^{-1}N_l(s)$ and $N_r(s)D_r(s)^{-1}$, where $D_l(s)$, $N_l(s)$, $N_r(s)$ and $D_r(s)$ are polynomial matrices of appropriate size. The matrix Routh algorithm (Shieh and Gaudiano 1974) and its reverse process (Shieh *et al.* 1975) of the algorithm for a multivariable system ($n_i = n_0$) are expressed as follows :

$$\left.\begin{array}{llll}
H_1 = A_{11}A_{21}^{-1} & A_{11} & A_{12} & A_{13} \ldots A_{1,n}A_{1,n+1} \\
 & A_{21} & A_{22} & A_{23} \ldots A_{2,n} \\
H_2 = A_{21}A_{31}^{-1} & & & \\
 & A_{31} \triangleq A_{12} - H_1A_{22} & A_{32} \triangleq A_{13} - H_1A_{23} & A_{33} \ldots \\
H_3 = A_{31}A_{41}^{-1} & & & \\
 & A_{41} \triangleq A_{22} - H_2A_{32} & A_{42} \triangleq A_{23} - H_2A_{33} & \cdot \ldots \\
\vdots & & & \\
 & A_{2n,1} & & \\
H_{2n} = A_{2n,1}A_{2n+1,1}^{-1} & & & \\
 & A_{2n+1,1} & &
\end{array}\right\} \quad (3\,a)$$

The H_j in eqn. (3 *a*) are the matrix quotients. The block elements of the first and second rows of eqn. (3 *a*) are the matrix coefficients of eqn. (1 *a*). The block elements of the subsequent rows are evaluated by the following matrix Routh algorithm :

$$\left.\begin{array}{l}
H_i = A_{i,1}A_{i+1,1}^{-1} \quad \text{for} \quad i = 1, 2, \ldots, 2k \quad \text{and} \quad k \leqslant n \\
\text{rank } A_{i+1,1} = n_i = n_0 \\
A_{i,j} = A_{i-2,j+1} - H_{i-2}A_{i-1,j+1} \quad \text{for} \quad j = 1, 2, \ldots; \quad i = 3, 4, \ldots
\end{array}\right\} \quad (3\,b)$$

When the two matrix polynomials $A_1(s)$ and $A_2(s)$ have no common factor, the matrix Routh array will terminate normally (i.e. we have $2n$ matrix quotients). When the two matrix polynomials have a common factor (the common divisor), the matrix Routh array in eqn. (3 *a*) will terminate prematurely, and the last non-vanishing row consists of the matrix coefficients of the common factor $B(s)$ in the original matrix polynomials $A_1(s)$ and $A_2(s)$. If we have $2k$ matrix quotients H_j, we can construct a pair of relatively prime matrix polynomials, $N_r(s)$ and $D_r(s)$, by using the reverse process of the matrix Routh algorithm in eqn. (3 *b*) :

$$\left.\begin{aligned} &P_{2k+1,1}=I \\ &P_{i,1}=H_iP_{i+1,1} \quad \text{for} \quad i=2k,\, 2k-1,\, \ldots,\, 2,\, 1 \\ &P_{i-2,j+1}=P_{i,j}+H_{i-2}P_{i-1,j+1} \quad \text{for} \quad i=2k+1,\, 2k,\, \ldots,\, 3\,; \quad j=1,\, 2,\, \ldots,\, k \end{aligned}\right\} \quad (3\,c)$$

The $T(s)$ in eqn. (1) is

$$\begin{aligned} T(s)&=A_2(s)A_1(s)^{-1}=N_r(s)B(s)[D_r(s)B(s)]^{-1}=N_r(s)D_r(s)^{-1} \\ &=[P_{21}+P_{22}s+\ldots+P_{2,k}s^{k-1}][P_{11}+P_{12}s+\ldots+P_{1,k+1}s^k]^{-1} \end{aligned} \quad (4)$$

The procedure can be well illustrated by the following numerical example.

Example 1

Consider that the common divisor $B(s)$ and a pair of relatively prime matrix polynomials $N_r(s)$ and $D_r(s)$ of the following matrix transfer function $T(s)$ are required :

$$T(s)=A_2(s)A_1(s)^{-1}=N_r(s)B(s)[D_r(s)B(s)]^{-1}=N_r(s)D_r(s)^{-1} \quad (5)$$

where

$$\begin{aligned} A_1(s)&=A_{11}+A_{12}s+A_{13}s^2+A_{14}s^3 \\ &=\begin{pmatrix}3 & 3\\ 1 & -3\end{pmatrix}+\begin{pmatrix}7 & 1\\ 1 & -5\end{pmatrix}s+\begin{pmatrix}5 & -3\\ 0 & -3\end{pmatrix}s^2+\begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix}s^3 \\ A_2(s)&=A_{21}+A_{22}s+A_{23}s^2 \\ &=\begin{pmatrix}6 & -2\\ -1 & -3\end{pmatrix}+\begin{pmatrix}6 & -6\\ -1 & 0\end{pmatrix}s+\begin{pmatrix}2 & -4\\ 0 & -1\end{pmatrix}s^2 \end{aligned}$$

$$n_0=n_i=2 \quad \text{and} \quad n=3$$

The matrix Routh array is

$$
\left.
\begin{aligned}
&H_1=\begin{pmatrix} 0\cdot3 & -1\cdot2 \\ 0\cdot3 & 0\cdot8 \end{pmatrix} \quad
A_{11}=\begin{pmatrix} 3 & 3 \\ 1 & -3 \end{pmatrix} \quad
A_{12}=\begin{pmatrix} 7 & 1 \\ 1 & -5 \end{pmatrix} \quad
A_{13}=\begin{pmatrix} 5 & -3 \\ 0 & -3 \end{pmatrix} \quad
A_{14}=\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \\
& \quad A_{21}=\begin{pmatrix} 6 & -2 \\ -1 & -3 \end{pmatrix} \quad
A_{22}=\begin{pmatrix} 6 & -6 \\ -1 & 0 \end{pmatrix} \quad
A_{23}=\begin{pmatrix} 2 & -4 \\ 0 & -1 \end{pmatrix} \\
&H_2=\begin{pmatrix} 1\cdot5 & 1\cdot9375 \\ -0\cdot25 & 0\cdot71875 \end{pmatrix} \quad
A_{31}=\begin{pmatrix} 4 & 2\cdot8 \\ 0 & -3\cdot2 \end{pmatrix} \quad
A_{32}=\begin{pmatrix} 4\cdot4 & -3 \\ -0\cdot6 & -1 \end{pmatrix} \quad
A_{33}=\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \\
&H_3=\begin{pmatrix} 6\cdot45 & 0\cdot7 \\ -6\cdot8 & 7\cdot2 \end{pmatrix} \quad
A_{41}=\begin{pmatrix} 0\cdot5625 & 0\cdot4375 \\ 0\cdot53125 & -0\cdot03125 \end{pmatrix} \quad
A_{42}=\begin{pmatrix} 0\cdot5 & -0\cdot5625 \\ 0\cdot25 & -0\cdot53125 \end{pmatrix} \\
&H_4=\begin{pmatrix} 0\cdot5 & 0\cdot0625 \\ 0\cdot25 & 0\cdot28125 \end{pmatrix} \quad
A_{51}=\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad
A_{52}=\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \\
& \quad A_{61}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}
\right\} \quad (6\,a)
$$

The matrix Routh array terminates prematurely because the only one block element A_{61} in the sixth row is a null matrix ; therefore, the common divisor $B(s)$ in $T(s)$ is

$$B(s) = A_{51} + A_{52}s = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} s \tag{6 b}$$

By using the matrix quotients $H_1 \dots H_4$ in eqn. (6 *a*) and applying the algorithm in eqn. (3 *c*) we have

$$\left.\begin{aligned} N_r(s) &= P_{21} + P_{22}s = \begin{pmatrix} 2 & 4 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} s \\ &\text{and} \\ D_r(s) &= P_{11} + P_{12}s + P_{13}s^2 = \begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ -1 & 3 \end{pmatrix} s + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} s^2 \end{aligned}\right\} \tag{6 c}$$

In order to show that the $B(s)$ in eqn. (6 *b*) is a common divisor of $A_1(s)$ and $A_2(s)$ in eqn. (5) we replace $A_{i,j}$ in eqn. (3 *b*) by P_{ij} in eqn. (6 *c*) and apply the algorithm in eqn. (3 *b*) to eqn. (6 *c*). Thus we have the following alternative matrix Routh array that has the same matrix quotients H_i as eqn. (6 *a*) :

$$
\left.
\begin{array}{lll}
H_1=\begin{pmatrix} 0\cdot 3 & -1\cdot 2 \\ 0\cdot 3 & 0\cdot 8 \end{pmatrix} & P_{11}=H_1H_2H_3H_4=\begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix} & P_{12}=H_1H_2+H_1H_4+H_3H_4=\begin{pmatrix} 4 & 0 \\ -1 & 3 \end{pmatrix} \quad P_{13}=I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
H_2=\begin{pmatrix} 1\cdot 5 & 1\cdot 9375 \\ -0\cdot 25 & 0\cdot 71875 \end{pmatrix} & P_{21}=H_2H_3H_4=\begin{pmatrix} 2 & 4 \\ -2 & 1 \end{pmatrix} & P_{22}=H_2+H_4=\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} \\
H_3=\begin{pmatrix} 6\cdot 45 & 0\cdot 7 \\ -6\cdot 8 & 7\cdot 2 \end{pmatrix} & P_{31}=H_3H_4=\begin{pmatrix} 3\cdot 4 & 0\cdot 6 \\ -1\cdot 6 & 1\cdot 6 \end{pmatrix} & P_{32}=I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
H_4=\begin{pmatrix} 0\cdot 5 & 0\cdot 0625 \\ 0\cdot 25 & 0\cdot 28125 \end{pmatrix} & P_{41}=H_4=\begin{pmatrix} 0\cdot 5 & 0\cdot 0625 \\ 0\cdot 25 & 0\cdot 28125 \end{pmatrix} & \\
 & P_{51}=I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &
\end{array}
\right\} \quad (6\,d)
$$

The justification for the $B(s)$ in eqn. (6 *b*), which is the common divisor of the matrix polynomials in eqn. (5), can be proved by the following induction method. Since the matrix Routh algorithm is developed from the repeated process of long division of two polynomial matrices, the reverse process of the algorithm can be applied to eqns. (6 *a*) and (6 *d*) to obtain the following identities :

$$\left.\begin{aligned}
A_{51}+A_{52}s &= P_{51}(A_{51}+A_{52}s)=P_{51}B(s)=B(s)\\
A_{41}+A_{42}s &= H_4(A_{51}+A_{52}s)+sA_{61}\\
&= H_4B(s)\\
&= P_{41}B(s)\\
A_{31}+A_{32}s+A_{33}s^2 &= H_3(A_{41}+A_{42}s)+s(A_{51}+A_{52}s)\\
&= H_3P_{41}B(s)+sP_{51}B(s)\\
&= (P_{31}+P_{32}s)B(s)\\
A_{21}+A_{22}s+A_{23}s^2 &= H_2(A_{31}+A_{32}s+A_{33}s^2)+s(A_{41}+A_{42}s)\\
&= H_2(P_{31}+P_{32}s)B(s)+sP_{41}B(s)\\
&= (P_{21}+P_{22}s)B(s)\\
A_{11}+A_{12}s+A_{13}s^2+A_{14}s^3 &= H_1(A_{21}+A_{22}s+A_{23}s^2)+s(A_{31}+A_{32}s+A_{33}s^2)\\
&= H_1(P_{21}+P_{22}s)B(s)+s(P_{31}+P_{32}s)B(s)\\
&= (P_{11}+P_{12}s+P_{13}s^2)B(s)
\end{aligned}\right\} \quad (6\,e)$$

From the last two equations in eqn. (6 *e*) we observe that

$$\left.\begin{aligned}
A_2(s) &= N_r(s)B(s)\\
A_1(s) &= D_r(s)B(s)
\end{aligned}\right\} \quad (6\,f)$$

Therefore $B(s)$ is the common divisor of the two polynomial matrices $A_1(s)$ and $A_2(s)$.

When $n_i \neq n_0$ and rank $A_{j,1} \neq q$ in eqn. (3 *b*), the matrix Routh algorithm in eqn. (3 *b*) cannot be directly applied. The matrix Routh algorithm and its reverse process of the algorithm in eqn. (3) are modified and discussed by the following case studies.

(1) $$T(s)=A_2(s)A_1(s)^{-1} \quad (7)$$

where

$$A_2(s)=\sum_{i=1}^{n} A_{2,i}s^{i-1} \quad \text{and} \quad A_1(s)=\sum_{i=1}^{n+1} A_{1,i}s^{i-1}$$

Case 1

$n_0 \geqslant n_i$, $T(s)\in R(s)^{n_0\times n_i}$, $A_2(s)\in R[s]^{n_0\times n_i}$, $A_1(s)\in R[s]^{n_i\times n_i}$

$$T(s)=A_2(s)A_1(s)^{-1}=N_r(s)B(s)[D_r(s)B(s)]^{-1}=N_r(s)D_r(s)^{-1} \quad (8\,a)$$

where

$$N_r(s)\in R[s]^{n_0\times n_i},\ N_r(s)=\sum_{i=1}^{k} P_{2,i}s^{i-1}\ ;\ D_r(s)\in R[s]^{n_i\times n_i}$$

$$D_r(s)=\sum_{i=1}^{k+1} P_{1,i}s^{i-1},\ P_{1,k+1}=I_{n_i}\ ;\ B(s)\in R[s]^{n_i\times n_i},\ B(s)=\sum_{i=1}^{n-k+1} B_is^{i-1}$$

The $B(s)$ is the common right divisor of the $A_2(s)$ and $A_1(s)$. For the use of the matrix Routh algorithm, the matrix coefficients in the $N_r(s)$ and $D_r(s)$ are expressed by the double-subscript notation as $P_{1,i}$ and $P_{2,i}$ which can be obtained by the algorithms as follows.

The matrix Routh algorithm is

$$\left.\begin{aligned} &H_i = A_{i,1}A_{i+1,1}^{-1}, \quad i=1,2,\ldots,2k \quad \text{and} \quad k \leqslant n \\ &\text{rank } A_{i,1} = n_i \\ &A_{i,j} = A_{i-2,j+1} - H_{i-2}A_{i-1,j+1}, \quad j=1,2,\ldots, \quad i=3,4,\ldots \end{aligned}\right\} \qquad (8\,b)$$

The constant matrices H_i with appropriate size are called the matrix quotients. If $n_0 > n_i$, the pseudo-inverse of $A_{i+1,1} = A_{i+1,1}^{-1} = [A_{i+1,1}^{\mathrm{T}} A_{i+1,1}]^{-1} A_{i+1,1}^{\mathrm{T}}$ is the left inverse of $A_{i+1,1}$.

The reverse process of the matrix Routh algorithm is

$$\left.\begin{aligned} &P_{2k+1,1} = I_{n_i} \\ &P_{l,1} = H_l P_{l+1,1}, \quad l=2k, 2k-1, \ldots, 2, 1 \\ &P_{i-2,j+1} = P_{i,j} + H_{i-2}P_{i-1,j+1}, \quad i=2k+1, 2k, \ldots, 3, \quad j=1,2,\ldots,k \end{aligned}\right\} \qquad (8\,c)$$

Case 2

$n_0 \leqslant n_i$, $T(s) \in R(s)^{n_0 \times n_i}$, $A_2(s) \in R[s]^{n_0 \times n_0}$, $A_1(s) \in R[s]^{n_i \times n_0}$

$$T(s) = A_2(s)A_1(s)^{-1} = D_l(s)^{-1}B(s)[N_l(s)^{-1}B(s)]^{-1} = D_l(s)^{-1}N_l(s) \qquad (9\,a)$$

where

$$N_l(s) \in R[s]^{n_0 \times n_i}, \; N_l(s) = \sum_{i=1}^{k} Q_{2,i}s^{i-1}; \quad D_l(s) \in R[s]^{n_0 \times n_0}$$

$$D_l(s) = \sum_{i=1}^{k+1} Q_{1,i}s^{i-1}, \; Q_{1,k+1} = I_{n_0}; \quad B(s) \in R[s]^{n_0 \times n_0}, \; B(s) = \sum_{i=1}^{n-k+1} B_i s^{i-1}$$

For the use of the matrix Routh algorithm, the matrix coefficients in the $N_l(s)$ and $D_l(s)$ are expressed by the double-subscript notation as $Q_{1,i}$ and $Q_{2,i}$ which can be obtained by the algorithms as follows.

The matrix Routh algorithm in eqn. (8 *b*) is applied to determine the matrix quotients H_i:

$$\left.\begin{aligned} &H_i = A_{i,1}A_{i+1,1}^{-1}, \quad i=1,2,\ldots,2k \quad \text{and} \quad k \leqslant n \\ &\text{rank } A_{i,1} = n_0 \\ &A_{i,j} = A_{i-2,j+1} - H_{i-2}A_{i-1,j+1}, \quad j=1,2,\ldots, \quad i=3,4,\ldots \end{aligned}\right\} \qquad (9\,b)$$

The new reverse algorithm is

$$\left.\begin{aligned} &Q_{2k+1,1} = I_{n_0} \\ &Q_{l,1} = Q_{l+1,1}H_l, \quad l=2k, 2k-1, \ldots, 2, 1 \\ &Q_{i-2,j+1} = Q_{i,j} + Q_{i-1,j+1}H_{i-2}, \quad i=2k+1, 2k, \ldots, 3, \quad j=1,2,\ldots,k \end{aligned}\right\} \qquad (9\,c)$$

(2) $$T(s)=A_1(s)^{-1}A_2(s) \tag{10}$$

where

$$A_2(s)=\sum_{i=1}^{n} A_{2,i}s^{i-1} \text{ and } A_1(s)=\sum_{i=1}^{n+1} A_{1,i}s^{i-1}.$$

Case 1

$$n_0 \leqslant n_i,\ T(s)\in R(s)^{n_0\times n_i},\ A_2(s)\in R[s]^{n_0\times n_i},\ A_1(s)\in R[s]^{n_0\times n_0}$$

$$T(s)=A_1(s)^{-1}A_2(s)=[B(s)D_l(s)]^{-1}B(s)N_l(s)=D_l(s)^{-1}N_l(s) \tag{11 a}$$

where

$$N_l(s)\in R[s]^{n_0\times n_i},\ N_l(s)=\sum_{i=1}^{k} Q_{2,i}s^{i-1}\ ;\ D_l(s)\in R[s]^{n_0\times n_0}$$

$$D_l(s)=\sum_{i=1}^{k+1} Q_{1,i}s^{i-1},\ Q_{1,k+1}=I_{n_0}\ ;\ B(s)\in R[s]^{n_0\times n_0},\ B(s)=\sum_{i=1}^{n-k+1} B_is^{i-1}$$

The matrix coefficients in the $D_l(s)$ and $N_l(s)$ are expressed by the double-subscript notation as $Q_{1,i}$ and $Q_{2,i}$ which can be determined by the following algorithms.

The new matrix Routh algorithm is

$$\left.\begin{aligned} &M_i=A_{i+1,1}{}^{-1}A_{i,1}, \quad i=1,2,\dots,2k \quad \text{and} \quad k\leqslant n \\ &\text{rank } A_{i,1}=n_0 \\ &A_{i,j}=A_{i-2,j+1}-A_{i-1,j+1}M_{i-2}, \quad j=1,2,\dots, \quad i=3,4,\dots \end{aligned}\right\} \tag{11 b}$$

The constant matrices M_i with appropriate size are called the matrix quotients. If $n_0<n_i$, $A_{i+1,1}{}^{-1}=A_{i+1,1}{}^{\mathrm{T}}[A_{i+1,1}A_{i+1,1}{}^{\mathrm{T}}]^{-1}$ is the right invesre of the $A_{i+1,1}$.

The reverse algorithm in (9 c) is applied to determine the $Q_{1,i}$ and $Q_{2,i}$.

$$\left.\begin{aligned} &Q_{2k+1,1}=I_{n_0} \\ &Q_{l,1}=Q_{l+1,1}M_l, \quad l=2k,2k-1,\dots,2,1 \\ &Q_{i-2,j+1}=Q_{i,j}+Q_{i-1,j+1}M_{i-2}, \quad i=2k+1,2k,\dots,3, \quad j=1,2,\dots,k \end{aligned}\right\} \tag{11 c}$$

Case 2

$$n_0 \geqslant n_i,\ T(s)\in R(s)^{n_0\times n_i},\ A_2(s)\in R[s]^{n_i\times n_i},\ A_1(s)\in R[s]^{n_i\times n_0}$$

$$T(s)=A_1(s)^{-1}A_2(s)=[B(s)N_r(s)^{-1}]^{-1}B(s)D_r(s)^{-1}=N_r(s)D_r(s)^{-1} \tag{12 a}$$

where

$$N_r(s)\in R[s]^{n_0\times n_i},\ N_r(s)=\sum_{i=1}^{k} P_{2,i}s^{i-1}\ ;\ D_r(s)\in R[s]^{n_i\times n_i}$$

$$D_r(s)=\sum_{i=1}^{k+1} P_{1,i}s^{i-1},\ P_{1,k+1}=I_{n_i}\ ;\ B(s)\in R[s]^{n_i\times n_i},\ B(s)=\sum_{i=1}^{n-k+1} B_is^{i-1}$$

The matrix coefficients in the $D_r(s)$ and $N_r(s)$ are expressed by the double-subscript notation as $P_{1,i}$ and $P_{2,i}$ which can be obtained by the algorithms as follows.

The matrix Routh algorithm in eqn. (11 *b*) is applied to determine the matrix quotients M_i :

$$\left.\begin{aligned} &M_i = A_{i+1,1}^{-1} A_{i,1}, \quad i=1,2,\ldots,2k \quad \text{and} \quad k \leqslant n \\ &\text{rank } A_{i,1} = n_i \\ &A_{i,j} = A_{i-2,j+1} - A_{i-1,j+1} M_{i-2}, \quad j=1,2,\ldots, \quad i=3,4,\ldots \end{aligned}\right\} \tag{12 b}$$

The reverse algorithm in eqn. (8 *c*) is applied to determine the $P_{1,i}$ and $P_{2,i}$:

$$\left.\begin{aligned} &P_{2k+1,1} = I_{n_i} \\ &P_{l,1} = M_l P_{l+1,1}, \quad l=2k, 2k-1, \ldots, 2, 1 \\ &P_{i-2,j+1} = P_{i,j} + M_{i-2} P_{i-1,j+1}, \quad i=2k+1, 2k, \ldots, 3, \quad j=1,2,\ldots,k \end{aligned}\right\} \tag{12 c}$$

By using Gilbert's theorem it has been shown (Shieh and Gaudiano 1975) that the dynamic state equations, which are constructed by using $2k$ matrix quotients H_i or M_i that are obtained from the matrix Routh algorithms, are minimal realizations of the $T(s)$. The minimal dimension of the system matrix is $kq \times kq$, where $q = \min(n_i, n_0)$ and $kq = \text{rank } T(s) \triangleq r_0$. The rank $T(s)$ can be determined from the Hankel matrix (Ho and Kalman 1966). By using the same $2k$ matrix quotients H_i or M_i and performing the reverse process of the matrix Routh algorithm, the monic polynomial matrices $D_r(s)$ and $D_l(s)$ are obtained and shown in eqns. (8), (9), (11) and (12). The highest power of s in the $D_r(s)$ and $D_l(s)$ is k and the matrix coefficients of s^k (i.e. $P_{1,k+1}$ and $Q_{1,k+1}$) are identity matrices having size $q \times q$. Therefore

$$\det D_r(s) = \det D_l(s) = \sum_{i=1}^{kq+1} d_i s^{i-1} \triangleq d(s) \tag{13}$$

The highest power of s in the monic polynomial $d(s)$ is kq, which is the rank of the $T(s)$. As a result, the $d(s)$ in eqn. (13) is the characteristic polynomial of the $T(s)$ and the polynomial matrices $D_r(s)$ and $N_r(s)$ are right co-prime and the $D_l(s)$ and $N_l(s)$ are left co-prime.

From the above discussion we also note that the necessary condition for the existence of the matrix Routh algorithm is that the ratio (denoted as k) of the rank $T(s)$ and the minimal dimension of the $T(s)$, $r_0/q = k$, is an integer. If the ratio r_0/q is not an integer or it is an integer but the condition (rank $A_{i,1} = q$) in the matrix Routh algorithm in eqns. (8)–(12) is violated due to the ill-conditioned matrix $A_{i,1}$, then the original $T(s)$ should be modified. $T(s)$ is modified by adding another transfer-function matrix $T_0(s) = 1/g(s)K$ whose rank is of $(kq - r_0)$, where k is the nearest integer and the scalar polynomial $g(s)$ is not a factor of the $\Delta_0(s)$ in eqn. (2). The K is a constant matrix with appropriate dimension. The modified system $T^1(s) \in R(s)^{n_0 \times n_i}$ is

$$T^1(s) = T(s) + \frac{1}{g(s)} K \tag{14}$$

where rank $[(1/g(s))K] = kq - r_0$ and rank $T^1(s) = kq$. It is noted that the addition of $(1/g(s))K$ to the $T(s)$ does not affect the locations of the poles of the $T(s)$.

3. Determination of poles and transmission zeros

By using the matrix Routh algorithm the $T(s)$ is factored into $D_l(s)^{-1}N_l(s)$ and $N_r(s)D_r(s)^{-1}$, where $D_l(s)$ and $N_l(s)$ are left co-prime and $N_r(s)$ and $D_r(s)$ are right co-prime. When $n_i = n_0 = q$, Desoer and Schulman (1974) have shown that the transmission zeros of the $T(s)$ are the zeros of the scalar polynomial $n(s)$, or

$$n(s) = \det N_l(s) = \det N_r(s) = 0 \tag{15}$$

where $N_l(s)$, $N_r(s)$, $D_l(s)$ and $D_r(s)$ are polynomial matrices ; $N_l(s)$ and $N_r(s)$ are $q \times q$; $D_l(s)$ and $D_r(s)$ are $q \times q$. The poles of the $T(s)$ are the zeros of the following characteristic equation :

$$\Delta(s) = \det D_l(s) = \det D_r(s) = 0 \tag{16}$$

When $r_0/q \neq k$ (an integer), the matrix Routh algorithm cannot be applied. The procedure shown in eqn. (14) can be applied to determine a pair of relatively prime polynomial matrices $D_l^1(s)$ and $N_l^1(s)$ or $N_r^1(s)$ and $D_r^1(s)$ as follows :

$$T^1(s) = D_l^1(s)^{-1}N_l^1(s) = N_r^1(s)D_r^1(s)^{-1} \tag{17 a}$$

The poles of the $T^1(s)$ can be determined from the following equations :

$$\det D_l^1(s) = \det D_r^1(s) = \{g(s)\}^{kq-r_0}P(s) = 0 \tag{17 b}$$

where the $g(s)$ is the polynomial used in eqn. (14). The poles of the $T(s)$ are the zeros of $P(s) = 0$.

When $r_0/q = k$ (an integer) and $n_0 \neq n_i$, the $N_l(s)$ and $N_r(s)$ obtained from the matrix Routh algorithm are not square polynomial matrices of size $n_0 \times n_0$ and $n_i \times n_i$. Therefore the transmission zeros cannot be directly determined from eqn. (15). The transmission zeros of the $T(s)$ can be determined from the invariant zeros of the determinants of all $q \times q$ minors of the $N_l(s)$ or $N_r(s)$ in eqn. (15) where $q = \min(n_0, n_i)$. However, when the n_i is much larger than the n_0 and vice versa, there exist many $q \times q$ minors which are expressed by polynomial matrices.

It is a cumbersome task to find the determinants of these $q \times q$ minors and to determine the roots of many polynomials. This difficulty can be overcome by applying Desoer and Schulman's (1974) theorem. The transmission zeros of the $T(s)$ are obtained from the invariant poles of two generalized inverses of the modified $T(s)$. In this paper we present a procedure to obtain the generalized inverses of the modified $T(s)$ so that the transmission zeros of the $T(s)$ can be determined. The steps are shown as follows.

Step 1. Modify the $T(s)$ and formulate the generalized inverses $T_i^*(s)$, $i = 1, 2$ of the modified $T(s)$, or

$$T_i^*(s) = [m_i(s)T(s)]^{-1} = A_1(s)[m_i(s)A_2(s)]^{-1}, \quad n_0 \geq n_i \tag{18 a}$$

$$= [m_i(s)T(s)]^{-1} = [m_i(s)A_2(s)]^{-1}A_1(s), \quad n_0 \leq n_i \tag{18 b}$$

where $T_i^*(s) \in R(s)^{n_i \times n_0}$, $A_1(s) \in R[s]^{q \times q}$, $A_2(s) \in R[s]^{n_0 \times n_i}$, $m_i(s) \in R[s]$. The monic polynomials $m_i(s) \in R[s]$, $i = 1, 2$, should not be the factors of the $\Delta_0(s)$ in eqn. (2) and of the $g(s)$ in eqn. (14), but are chosen in such a way that the power of the

s in the polynomial matrices $m_i(s)A_2(s)$ in eqn. (18) is 1° higher than that of the $A_1(s)$. This modification does not affect the transmission zeros of the $T(s)$ but it makes the matrix Routh algorithm applicable.

Step 2. Modify the $T_i^*(s)$ when $r_0/q \neq k$ (an integer).

The process of this modification is shown in eqn. (14). The additions of $(1/g(s))K$, $i=1, 2$ to the $T_i^*(s)$, $i=1, 2$, do not affect the locations of the poles of the $T_i^*(s)$ or the transmission zeros of the $T(s)$. After performing some matrix operations we have the generalized inverses of the modified $T(s)$ denoted as $T_i^+(s)$, $i=1, 2$:

$$T_i^+(s) = [g_i(s)A_1(s) + m_i(s)KA_2(s)][g_i(s)m_i(s)A_2(s)]^{-1}, \quad n_0 \geqslant n_i \tag{19 a}$$

$$= [g_i(s)m_i(s)A_2(s)]^{-1}[g_i(s)A_1(s) + m_i(s)A_2(s)K], \quad n_0 \leqslant n_i \tag{19 b}$$

Step 3. Determine two pairs of relatively prime polynomial matrices.

The algorithms in eqns. (9) and (12) can be applied to obtain two pairs of left co-prime polynomial matrices denoted as $D_{li}^*(s)$ and $N_{li}^*(s)$, $i=1, 2$ or right co-prime polynomial matrices $N_{ri}^*(s)$ and $D_{ri}^*(s)$, $i=1, 2$:

$$T_i^+(s) = D_{li}^*(s)^{-1}N_{li}^*(s), \quad n_0 \geqslant n_i \tag{20 a}$$

$$= N_{ri}^*(s)D_{ri}^*(s)^{-1}, \quad n_0 \leqslant n_i \tag{20 b}$$

Step 4. Select the required transmission zeros.

The poles of the $T_i^+(s)$, $i=1, 2$, are

$$\det D_{li}^*(s) = \{m_i(s)\}^q n(s)k_i(s) = 0, \quad n_0 \geqslant n_i \tag{21 a}$$

$$\det D_{ri}^*(s) = \{m_i(s)\}^q n(s)k_i(s) = 0, \quad n_0 \leqslant n_i \tag{21 b}$$

The required transmission zeros of the $T(s)$ are the invariant poles of the $T_i^+(s)$, $i=1, 2$. They are the zeros of polynomial $n(s)$ in eqn. (21), or

$$n(s) = 0 \tag{21 c}$$

When the $T(s)$ is not a strictly proper rational matrix transfer function, eqn. (19) can also be used to determine the poles and transmission zeros of the $T(s)$.

Example 2

Consider that the poles and transmission zeros of the following matrix transfer function $T(s)$ are required to be determined :

$$T(s) = A_2(s)A_1(s)^{-1} \tag{22}$$

where

$$A_2(s) = \begin{bmatrix} s^3+6s^2+11s+12 & s^2+9s+20 \\ s^2+4s-5 & s^3+3s^2-7s+15 \\ s^3+6s^2+13s+10 & 3s^2+5s+26 \end{bmatrix}$$

and

$$A_1(s)=\begin{bmatrix} s^4+5s^3+11s^2+13s+6 & s^3+8s^2+17s+10 \\ s^3+2s^2-s-2 & s^4+s^3-s^2+5s+6 \end{bmatrix}$$

$$n_0=3 \quad \text{and} \quad n_i=2$$

It is difficult to apply most time-domain approaches to this problem. By using the proposed algorithms in eqn. (8), the $T(s)$ can be factored into a pair of relatively prime polynomial matrices $N_r(s)$ and $D_r(s)$:

$$T(s)=A_2(s)A_1(s)^{-1}=N_r(s)B(s)[D_r(s)B(s)]^{-1}=N_r(s)D_r(s)^{-1} \tag{23}$$

where

$$N_r(s)=\begin{bmatrix} s+4 & 0 \\ 0 & s+5 \\ s+4 & 2 \end{bmatrix} \quad \text{and} \quad D_r(s)=\begin{bmatrix} s^2+3s+2 & 0 \\ 0 & s^2+3s+2 \end{bmatrix}$$

Following eqn. (16) we have the required poles of the $T(s)$:

$$\Delta(s)=\det D_r(s)=(s+1)^2(s+2)^2=0 \tag{24 a}$$

or

$$s_1=s_2=-1 \quad \text{and} \quad s_3=s_4=-2 \tag{24 b}$$

It is interesting to note that the common factor $B(s)$ in eqn. (23) is

$$B(s)=\begin{bmatrix} s^2+2s+3 & s+5 \\ s-1 & s^2-2s-2 \end{bmatrix} \tag{25}$$

Since $n_0>n_i$ and the determination of the transmission zeros of the $T(s)$ are required, we construct the generalized inverses $T_i^*(s)$ in eqn. (18) from the modified $T(s)$ in eqn. (23), and apply the proposed algorithm in eqn. (9) to decompose the $T_i^*(s)$ into two pairs of left co-prime polynomial matrices $D_{li}^*(s)$ and $N_{li}^*(s)$, $i=1, 2$ in eqn. (20 a) as follows :

$$T_1^*(s)=A_1(s)[m_1(s)A_2(s)]^{-1}=D_r(s)[m_1(s)N_r(s)]^{-1}=D_{l1}^*(s)^{-1}N_{l1}^*(s) \tag{26 a}$$

where $m_1(s)=s^2+s+1$ is not a factor of the $\Delta(s)$ in eqn. (24 a) :

$$D_{l1}^*(s)=\begin{bmatrix} s^3+5s^2+5s+4 & 1{\cdot}1965s^2+2{\cdot}1649s+0{\cdot}81525 \\ 0 & s^3+5{\cdot}88073s^2+5{\cdot}88073s+4{\cdot}88073 \end{bmatrix}$$

$$N_{l1}^*(s)=\begin{bmatrix} 0{\cdot}4275s^2+1{\cdot}329s+0{\cdot}981 & 0{\cdot}0177s+0{\cdot}057 & 0{\cdot}573s^2+1{\cdot}671s+1{\cdot}02 \\ 0{\cdot}0797s^2+0{\cdot}1407s-0{\cdot}574 & s^2+3{\cdot}04s+1{\cdot}72 & -0{\cdot}08s^2-0{\cdot}143s+0{\cdot}574 \end{bmatrix}$$

$$T_2^*(s)=A_1(s)[m_2(s)A_2(s)]^{-1}=D_r(s)[m_2(s)N_r(s)]^{-1}=D_{l2}^*(s)^{-1}N_{l2}^*(s) \tag{26 b}$$

where $m_2(s) = s^2 - 10$ is not a factor of the $\Delta(s)$ in eqn. (24 *a*) :

$$D_{l2}^*(s) = \begin{bmatrix} s^3 + 4s^2 - 10s - 40 & -3{\cdot}854672s^2 + 38{\cdot}54672 \\ 0 & s^3 + 0{\cdot}4438694s^2 - 10s - 4{\cdot}438694 \end{bmatrix}$$

$$N_{l2}^*(s) = \begin{bmatrix} 2{\cdot}24s^2 + 4{\cdot}45s + 1{\cdot}41 & -1{\cdot}38s - 1{\cdot}78 & -1{\cdot}24s^2 - 1{\cdot}45s + 0{\cdot}59 \\ 1{\cdot}02s^2 - 0{\cdot}36s - 0{\cdot}04 & s^2 + 0{\cdot}49s + 0{\cdot}16 & -1{\cdot}02s^2 + 0{\cdot}36s + 0{\cdot}038 \end{bmatrix}$$

Following eqn. (21 *a*) yields

$$\det D_{l1}^*(s) = (s^2 + s + 1)^2(s+4)(s + 4{\cdot}88073) = m_1^2(s)n(s)k_1(s) = 0 \qquad (27\ a)$$

$$\det D_{l2}^*(s) = (s^2 - 10)^2(s+4)(s + 0{\cdot}44386943) = m_2^2(s)n(s)k_2(s) = 0 \qquad (27\ b)$$

The common divisor $n(s)$ of the det $D_{l1}^*(s)$ and det $D_{l2}^*(s)$ is the common factor $n(s) \in R[s]$ in eqns. (27 *a*) and (27 *b*). The transmission zeros of the $T(s)$ which are the invariant poles of the $T_i^*(s)$ in eqn. (26) are the zeros of the polynomial $n(s)$, or

$$n(s) = s + 4 = 0 \qquad (28)$$

The transmission zero is $s = -4$.

The computation involves only arithmetic operations of small-size matrices. Therefore, it is believed that the proposed method is computationally superior to most time-domain approaches if the system is given in the frequency domain.

4. Conclusion

A purely algebraic method has been presented for factorizing a rational matrix transfer function into a pair of relatively prime polynomial matrices and for determining the poles and transmission zeros of a multivariable system. Also, the common divisor of two matrix polynomials can be determined from the matrix Routh algorithm and the matrix Routh array. When a matrix transfer function that might have a high degree common divisor is given, the method proposed in this paper is computationally superior to most time-domain methods because the proposed algorithm only deals with arithmetic operations of small-size matrices. The matrix Routh algorithm has been extended for general cases ($n_i \neq n_0$ and $n_i = n_0$).

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Some Sufficient and Some Necessary Conditions for the Stability of Multivariable Systems

Some sufficient and some necessary conditions for the stability of a class of multivariable systems represented by matrix polynomials are derived. A new linear block transformation is also established for transforming an observable block companion form to the block Schwarz form.

I Introduction

The accurate description of most practical systems, for example both a small semiactive terminal homing missile system [1] and an aircraft system [2], result in high order coupled multivariable differential equations. Linear representations of these systems are by a set of coupled high-order differential equations or a matrix differential equation. A primary concern in the design of these multivariable systems is the stability problem. One conventional approach is to formulate the system into a high dimensional state equation, then to determine the stability by either directly evaluating the roots of the scalar characteristic polynomial, indirectly applying the Routh criterion [3], or application of Jury's inner theory [4] on the characteristic polynomial. However, the determination of a characteristic polynomial for a large dimensional system is tedious. Moreover, if a system is modeled as a matrix differential equation, it is more natural to determine the stability directly from the matrix polynomial than indirectly from a scalar polynomial. Some approaches have been proposed to determine the stability of a multivariable system directly from the matrix polynomial. Papaconstantinou [5] suggested a scheme for testing stability of polynomial matrices. In his work, a recursive algorithm was developed to compare the normalized largest eigenvalues with unity. However, the method requires the calculation of the eigenvalues of largest moduli for indirectly determining the stability of polynomial matrices. Recently, Shieh and Sacheti [6] partially extended the scalar Routh criterion [3] to the matrix case. In this work, it is shown that, if a matrix polynomial $B(s) = Is^n + B_n s^{n-1} + \ldots + B_1$ is given, a matrix Routh array can be constructed by using the following matrix Routh algorithm:

$$C_{1,j} = B_{n+3-2j} \qquad j = 1, 2, 3, \ldots, l$$

$$\text{where } l = \begin{cases} \dfrac{n}{2} + 1 & n \text{ even} \\[2mm] \dfrac{n+1}{2} & n \text{ odd} \end{cases}$$

$$C_{2,j} = B_{n+2-2j} \qquad j = 1, 2, 3, \ldots, l$$

$$C_{11} = I$$

$$C_{i,j} = C_{i-2,j+1} - H_{i-2}C_{i-1,j+1} \qquad i = 1, 2, \ldots, \quad j = 3, 4, \ldots$$

$$H_i = C_{i,1}(C_{i+1,1})^{-1} \qquad i = 1, 2, \ldots, n$$

$$\det (C_{i+1,1}) \neq 0 \tag{1}$$

A sufficient condition for stability of the det $[B(s)]$ is that all the "matrix quotients" H_i be real, symmetric, positive definite matrices. Note that this sufficient condition deals only with H_i and not $C_{j,1}$ (the block elements in the first column of the matrix Routh array). Liapunov theory with the state equation in the controllable block companion (controllable phase-variable) form was used to derive their result.

In this paper, we develop two approaches for determining the stability of a class of multivariable systems. One approach uses the "matrix quotients" M_j that are developed from an alternate matrix Routh algorithm and a state equation in the observable block companion form [7]. The other approach uses the block elements in the first column of the matrix Routh array. Two sufficient conditions and three necessary conditions are derived for the stability of matrix polynomials, thereby partially extending the scalar Routh criterion to the matrix Routh criterion.

II Sufficient Conditions

The objective of this paper is to establish the criteria for the stability of the following matrix differential equations.

$$\sum_{i=1}^{n+1} B_i D^{i-1} x(t) = [0], \; B_{n+1} = I \tag{2a}$$

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and

$$D^{i-1}x(0) = [\alpha_{i-1}] \qquad i = 1, 2, 3, \ldots, n \tag{2b}$$

where $x(t)$ is the m-dimensional state vector. B_i, I, and [0] are $m \times m$ real constant matrix, identity matrix and null matrix, respectively. For the scalar case, it is well known that a system is asymptotically stable if and only if the Routh array elements in the first column are all positive. Shieh and Sacheti [6] partially extended the Routh criteria [3] to the matrix case and derived a sufficient condition for the stability of a multivariable system in equation (2) from the controllable block companion form. In this paper we derive some sufficient and some necessary conditions for the system in equation (2) from the observable block companion form.

Let us rewrite the system in equation (2) into the following observable block companion form:

$$[\dot{x}] = [B][x] \tag{3a}$$

$$[x(0)] = [\alpha] \tag{3b}$$

where

$$[B] = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & -B_1 \\ \mathrm{I} & 0 & 0 & \cdot & 0 & -B_2 \\ 0 & \mathrm{I} & 0 & \cdot & 0 & -B_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \mathrm{I} & -B_n \end{bmatrix}$$

The dimensions of the matrix $[B]$, the block elements B_i, and state vector $[x]$ are $(nm) \times (nm)$, $m \times m$, and $(nm) \times 1$, respectively. Equation (3) can be transformed into the block Schwarz form by using the following linear transformation:

$$[x] = [K_1][y] \tag{4a}$$

and

$$[\dot{y}] = [K_1]^{-1}[B][K_1][y] = [A][y] \tag{4b}$$

where

$$[K_1] = \begin{bmatrix} \mathrm{I} & \cdot & D_{n-1,2}D_{n-1,1}^{-1} & \cdot & D_{n-3,3}D_{n-3,1}^{-1} & 0 & D_{n-5,4}D_{n-5,1}^{-1} & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \mathrm{I} & \cdot & D_{62}D_{61}^{-1} & 0 & D_{43}D_{41}^{-1} & 0 & D_{24}D_{21}^{-1} \\ 0 & \cdot & 0 & \cdot & 0 & D_{52}D_{51}^{-1} & 0 & D_{33}D_{31}^{-1} & 0 \\ 0 & \cdot & 0 & \cdot & \mathrm{I} & 0 & D_{42}D_{41}^{-1} & 0 & D_{23}D_{21}^{-1} \\ 0 & \cdot & 0 & \cdot & 0 & \mathrm{I} & 0 & D_{32}D_{31}^{-1} & 0 \\ 0 & \cdot & 0 & \cdot & 0 & 0 & \mathrm{I} & 0 & D_{22}D_{21}^{-1} \\ 0 & \cdot & 0 & \cdot & 0 & 0 & 0 & \mathrm{I} & 0 \\ 0 & \cdot & 0 & \cdot & 0 & 0 & 0 & 0 & \mathrm{I} \end{bmatrix} \tag{4c}$$

and

$$[A] = \begin{bmatrix} 0 & -A_1 & 0 & \cdot & 0 & 0 \\ \mathrm{I} & 0 & -A_2 & \cdot & 0 & 0 \\ 0 & \mathrm{I} & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & -A_{n-1} \\ 0 & 0 & 0 & \cdot & \mathrm{I} & -A_n \end{bmatrix} \tag{4d}$$

The dimension of each block element in $[A]$ and $[K_1]$ is $m \times m$. The block elements $D_{i,j}$, having dimension $m \times m$, in equation (4c) can be obtained from the following alternate matrix Routh algorithm and alternate matrix Routh array which are different from those in equation (1).

Let us define $l = n/2 + 1$ if n is an even number, otherwise $l = (n + 1)/2$, and $D_{i,j}$ as follows:

$$D_{1,j} = B_{n+3-2j} \qquad j = 1, 2, 3, \ldots, l$$

$$D_{2,j} = B_{n+2-2j} \qquad j = 1, 2, 3, \ldots, l$$

$$D_{11} = \mathrm{I} \tag{5a}$$

The alternate matrix Routh array and the matrix Routh algorithm are:

$$\begin{array}{llll}
 & D_{11} = B_{n+1} & D_{12} = B_{n-1} & D_{13} = B_{n-3} \ldots \\
M_1 = D_{21}^{-1}D_{11} < & & & \\
 & D_{21} = B_n & D_{22} = B_{n-2} & D_{23} = B_{n-4} \ldots \\
M_2 = D_{31}^{-1}D_{21} < & & & \\
 & D_{31} \triangleq D_{12} - D_{22}M_1 & D_{32} \triangleq D_{13} - D_{23}M_1 & D_{33} \;.\;\; . \\
M_3 = D_{41}^{-1}D_{31} < & & & \\
 & D_{41} \triangleq D_{22} - D_{32}M_2 & D_{42} \triangleq D_{23} - D_{33}M_2 & D_{43} \;.\;\; . \\
M_4 = D_{51}^{-1}D_{41} < & & & \\
 & D_{51} \triangleq D_{32} - D_{42}M_3 & D_{52} & \\
 & \cdot \qquad \cdot & \cdot \qquad \cdot & \\
 & D_{n,1} & & \\
M_n = D_{n+1,1}^{-1}D_{n,1} < & & & \\
 & D_{n+1,1} & &
\end{array} \tag{5b}$$

where

$$D_{i,j} = D_{i-2,j+1} - D_{i-1,j+1}M_{i-2} \quad j = 1, 2, \ldots, \quad i = 3, 4, \ldots$$

$$M_i = D_{i+1,1}^{-1}D_{i,1} \qquad i = 1, 2, \ldots, n$$

$$\det [D_{i+1,1}] \neq 0 \tag{5c}$$

The construction of the matrix Routh array in equation (5b) is as follows. Arrange the matrix coefficients of the given matrix polynomial in equation (2a) in the first two rows of the array shown in equation (5b). A new matrix M_1 is obtained by the matrix multiplication $D_{21}^{-1}D_{11}$ where D_{11} and D_{21} are the block elements in the first column of the array. The block elements in the third row are generated from the M_1 and the block elements in the first two rows as follows: First, each block element in the second row is postmultiplied by M_1, then, subtract each resulting matrix from each block element in the first row, finally, shift each block element so obtained one column left and drop the zero-first block element to form the third row. The second and the obtained third row are then used as starting rows to generate the new matrix M_2 and the block elements in the fourth row. Repeating the processes to the $n + 1$ row yields the complete matrix Routh array. When any $D_{i+1,1}$ matrices other than D_{11} or $D_{n+1,1}$ in equation (5c) are singular, another set of $D_{i+1,1}$ can be chosen from the new matrix polynomial that is the product of the original matrix polynomial and an asymptotically stable matrix polynomial. Thus a new matrix Routh array can be obtained and the stability of the original matrix polynomial is preserved because the stability of the original matrix polynomial is invariant under this transformation. Shieh and Sacheti [6] have shown that if $H_i = C_{i,1}C_{i+1,1}^{-1}$ for $i = 1, 2, \ldots, n$ in equation (1) are positive definite, then the system in equation (2) is stable.

Here, we show similar results when replacing H_i by M_i. Note that a positive definite matrix means a matrix is real, symmetric and positive definite.

Theorem 1. If $\{M_i\}$ $i = 1, 2, \ldots, n$ in equation (5) are positive definite, then the system in equation (2) is stable.

Proof. Performing the following new transformation

$$[y] = [K_2][z] \tag{6}$$

on equation (4) yields

$$\begin{aligned} [\dot{z}] &= [K_2]^{-1}[A][K_2][z] \\ &= [F][z] \end{aligned} \tag{7a}$$

where

$$[K_2] = \begin{bmatrix} D_{n,1} & 0 & \cdot & 0 & 0 \\ 0 & D_{n-1,1} & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & D_{21} & 0 \\ 0 & 0 & \cdot & 0 & D_{11} \end{bmatrix} \tag{7b}$$

and

$$[F] = \begin{bmatrix} 0 & -M_n^{-1} & 0 & \cdot & 0 & 0 & 0 & 0 \\ M_{n-1}^{-1} & 0 & -M_{n-1}^{-1} & \cdot & 0 & 0 & 0 & 0 \\ 0 & M_{n-2}^{-1} & 0 & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & -M_4^{-1} & 0 & 0 \\ 0 & 0 & 0 & \cdot & M_3^{-1} & 0 & -M_3^{-1} & 0 \\ 0 & 0 & 0 & \cdot & 0 & M_2^{-1} & 0 & -M_2^{-1} \\ 0 & 0 & 0 & \cdot & 0 & 0 & M_1^{-1} & -M_1^{-1} \end{bmatrix} \tag{7c}$$

It is noticed that, if each block element in the matrix $[F]$ in equation (7c) were a scalar, then the matrix $[F]$ would be a matrix of the Schwarz form [8]. Since the elements are blocks, the matrix $[F]$ in equation (7c) is a block Schwarz form matrix.

The linear transformation matrix $[K]$ between x coordinates and z coordinates is

$$[x] = [K][z] = [K_1][K_2][z] \tag{7d}$$

Now, consider the following quadratic equation:

$$V = [z]^T[P][z] \tag{8a}$$

where

$$P = \begin{bmatrix} M_n & 0 & \cdot & 0 & 0 \\ 0 & M_{n-1} & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & M_2 & 0 \\ 0 & 0 & \cdot & 0 & M_1 \end{bmatrix} \tag{8b}$$

and T in equation (8a) designates transpose.

Since $\{M_i\}$ are positive definite which implies that P is positive definite, V is positive definite. The derivative of V is

$$\begin{aligned} \dot{V} &= [z]^T[PF + F^TP][z] \\ &= -[z]^T[Q][z] = -[z]^T[RR^T][z] \end{aligned} \tag{9a}$$

where

$$[Q] = \begin{bmatrix} 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 & 2I \end{bmatrix}, \quad [R] = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ \sqrt{2I} \end{bmatrix} \tag{9b}$$

rank $[Q]$ = rank $[R] = m$. From equations (8) and (9) we can see that V is a Liapunov function. Hence, we conclude that the system in equation (2) is stable.

From the result obtained in Theorem 1, we establish another sufficient condition for the stability of the system in equation (2) by using the block elements $D_{i,1}$ in the matrix Routh array in equation (5) instead of the M_i in equation (5).

Theorem 2. If $\{D_{i,1}\}$ $i = 2, 4, 6, \ldots$, are positive definite, the eigenvalues of $\{D_{i,1}\}$ $i = 1, 3, 5, \ldots$, are positive and real, and $\{D_{i,1}D_{i+1,1}\}$ $i = 1, 3, 5, \ldots$, $\{D_{i+1,1}D_{i,1}\}$ $i = 2, 4, 6, \ldots$, are Hermitian, the system [equation (2)] is stable.

In order to prove Theorem 2, we need the following lemma which is due to KyFan [9] [p. 137].

Lemma 1. Let K_1 be positive definite and K_2 such that K_1K_2 is Hermitian. Then K_1K_2 is positive definite if and only if the eigenvalues of K_2 are positive and real. In the following lemma, we switch the conditions on K_1 and K_2 yielding the same result.

Lemma 2. Let K_2 be positive definite and K_1 such that K_1K_2 is Hermitian. The K_1K_2 is positive definite if and only if the eigenvalues of K_1 are positive and real.

Proof. Since K_2 is positive definite which implies K_2^T is positive definite, where T designates transpose, it is seen from lemma 1 that $K_2^TK_1^T$ is positive definite if and only if the eigenvalues of K_1^T are positive and real. But $K_2^TK_1^T = (K_1K_2)^T$; i.e., K_1K_2 is positive definite if and only if the eigenvalues of K_1 are positive and real.

Lemma 3. If K_2 is positive definite and K_1K_2 is symmetric, then $K_1^{-1}K_2$ is symmetric.

Proof. Since $(K_1K_2)^T = K_2^TK_1^T = K_2K_1^T = K_1K_2$ which implies $K_1^T = K_2^{-1}K_1K_2$. Hence $(K_1^{-1}K_2)^T = K_2^T(K_1^{-1})^T = K_2K_2^{-1}K_1^{-1}K_2 = K_1^{-1}K_2$; i.e., $K_1^{-1}K_2$ is symmetric.

Proof of Theorem 2. By lemma 3, we know that $D_{i+1,1}^{-1}D_{i,1}$ is symmetric for $i = 1, 2, \ldots$. By lemma 2 or 3, we know that $M_i = D_{i+1,1}^{-1}D_{i,1}$ is positive definite. Hence, the system in equation (2) is stable following the results of Theorem 1.

In order to show an application of Theorem 1 and Theorem 2, let us consider the following matrix characteristic equation:

$$As^2 + Bs + C = 0 \tag{10a}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 2 \end{bmatrix}$$

If we arrange the matrices A, B, and C in equation (10a) by following the matrix Routh algorithm of equation (1), we obtain

$$\begin{aligned}
&C_{11} = A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C_{12} = C = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 2 \end{bmatrix} \\
&H_1 = \frac{1}{8}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} < \\
&C_{21} = B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \\
&H_2 = \frac{1}{2}\begin{bmatrix} 6 & 0 \\ 2 & \frac{8}{3} \end{bmatrix} < \\
&C_{31} = C = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 2 \end{bmatrix}
\end{aligned} \tag{10b}$$

In this case, no conclusion can be drawn from the sufficient condition established by Shieh and Sacheti [6]. However, if we arranged the matrices A, B, and C according to equation (5), we have

$$\begin{aligned}
&D_{11} = A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D_{12} = C = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 2 \end{bmatrix} \\
&M_1 = \frac{1}{8}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} < \\
&D_{21} = B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \\
&M_2 = \frac{1}{2}\begin{bmatrix} \frac{17}{3} & 1 \\ 1 & 3 \end{bmatrix} < \\
&D_{31} = C = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 2 \end{bmatrix}
\end{aligned} \tag{10c}$$

From Theorem 1, we see that the system is stable.

This example shows the application of Theorem 2. Let us consider the following matrix characteristic equation:

$$A_1s^2 + B_1s + C_1 = 0 \tag{11a}$$

where

$$D_{11} = A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{21} = B_1 = \begin{bmatrix} 25.77 & 13.7 \\ 13.7 & 7.3 \end{bmatrix},$$

$$D_{12} = C_1 = D_{31} = \begin{bmatrix} -1 & 2.1 \\ -1 & 2 \end{bmatrix}$$

In Bellman [9], [p. 67, p. 101] it is shown that, if A_1, B_1, and C_1 are positive definite, then the roots of $\det[A_1s^2 + B_1s + C_1] = 0$ have negative real parts. But in this example, no conclusion can be made from Bellman's results. However, we know that

$$D_{11} \cdot D_{21} = A_1 \cdot B_1 = \begin{bmatrix} 25.77 & 13.7 \\ 13.7 & 7.3 \end{bmatrix}$$

and

$$D_{21} \cdot D_{31} = C_1 \cdot B_1 = \begin{bmatrix} 3 & 1.63 \\ 1.63 & 0.9 \end{bmatrix} \tag{11b}$$

which are symmetric, B_1 is positive definite, and the eigenvalues of A_1 and C_1 are positive and real. Therefore, from Theorem 2 we conclude that the system in equation (11a) is stable. Although only second order matrix polynomials with 2×2 matrix coefficients are illustrated in the examples, the theory is valid for high order matrix polynomials.

III Necessary Conditions

In this section we establish some necessary conditions for the stability of multivariable systems. The failure to satisfy the necessary conditions for stability is equivalent to the sufficient conditions for the instability of the same systems; i.e.,

Theorem 3. If $\{M_i\}$ $i = 1, 2, \ldots, n$ are symmetric such that, there exists one $\{M_i\}$ $i = 1, 2, \ldots, n$ which is negative definite, negative semi-definite, or indefinite, then the system in equation (2) is unstable.

Proof. Suppose the system is asymptotically stable and one of $\{M_i\}$ is negative definite, negative semi-definite, or indefinite. Since the stability is invariant under the linear transformation and the matrix F in equation (7) is a stable matrix. Let us consider the following equation:

$$XF + F^TX = -Q \tag{12}$$

where Q is a matrix defined in equation (9*b*). By Thoerem 4 in Bellman [9] [p. 239] and the theorems in Anderson [10] and Barnett [11] [p. 86], we know that equation (12) has a unique solution. Since Q is positive semi-definite and rank $[Q]$ = rank $[R] = m$, we conclude that the solution X of equation (12) is also positive semidefinite. Furthermore, X is positive definite if the pair $[F, R^T]$ is observable. It is easy to verify that the matrix P which was defined in equation (8*b*) satisfies equation (12). Therefore $X = P$ is positive semidefinite or positive definite. This implies that at least one of the $\{M_i\}$ is positive semidefinite and others positive definite or all positive definite. This contradicts our assumption that one of the $\{M_i\}$ is negative definite, negative semidefinite, or indefinite. Hence the system in equation (2) is unstable if one of the $\{M_i\}$ is negative definite, negative semidefinite, or indefinite.

To show an application of Theorem 3, consider the example [5]:

$$A_1 \frac{d^2y}{dt^2} + B_1 \frac{dy}{dt} + C_1 y = 0 \tag{13a}$$

where

$$A_1 = \begin{bmatrix} 5 & 1 \\ 1 & 10 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 49 & 0 \\ 0 & 49 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -5 & -1 \\ -1 & -10 \end{bmatrix}$$

Applying equation (5) yields the matrix Routh array and M_j

$$D_{11} = A_1 = \begin{bmatrix} 5 & 1 \\ 1 & 10 \end{bmatrix} \qquad D_{12} = C_1 = \begin{bmatrix} -5 & -1 \\ -1 & -10 \end{bmatrix}$$

$$M_1 = \frac{1}{49}\begin{bmatrix} 5 & 1 \\ 1 & 10 \end{bmatrix} <$$

$$D_{21} = B_1 = \begin{bmatrix} 49 & 0 \\ 0 & 49 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} -10 & 1 \\ 1 & -5 \end{bmatrix} <$$

$$D_{31} = C_1 = \begin{bmatrix} -5 & -1 \\ -1 & -10 \end{bmatrix} \tag{13b}$$

M_2 is symmetric and indefinite. According to Theorem 3, the system is unstable.

The following theorem is another criteria for an unstable system.

Theorem 4. If $D_{11} = B_{n+1} = \mathrm{I}$ and the trace of D_{21} $(= B_n)$ is negative, then the system in equation (2) is unstable, where D_{11} and D_{21} are defined in equation (5*a*).

Proof. The matrix characteristic equation of the system in equation (2) is

$$\begin{aligned}[D(s)] &= D_{11}s^n + D_{21}s^{n-1} + D_{12}s^{n-2} + \ldots + D_{i,n} \\ &= B_{n+1}s^n + B_ns^{n-1} + B_{n-1}s^{n-2} + \ldots + B_1 \end{aligned} \tag{14}$$

where $i = 1$ if n is even and $i = 2$ if n is odd.

Since the sum of the eigenvalues of the system in equation (2) is equal to the negative value of the trace of D_{21}, this implies that there exists some eigenvalues of det $[D(s)]$ which are positive. Hence, the system is unstable.

The next criteria is another necessary condition which we state as follows.

Theorem 5. If det $D_{11} > 0$ and det $D_{i,n} < 0$, or det $D_{11} < 0$ and det $D_{i,n} > 0$, and D_{11}, $D_{i,n}$ are defined in equation (14), then the system in equation (2) is unstable.

Proof. Since the system in equation (2) has the matrix characteristic equation $[D(s)]$ in equation (14), then we expand the *det* $[D(s)]$. We find the constant term is equal to det B_1 = *det* $D_{i,n}$. If *det* $D_{11} > 0$ and *det* $D_{i,n} < 0$, this implies that the coefficient of the polynomial det $[D(s)]$ has a negative sign. We can then conclude that the det $[D(s)] = 0$ has a solution with a positive real part. Hence the system is unstable.

IV Conclusion

Some necessary and some sufficient conditions have been developed for the stability of a class of multivariable systems. A linear block transformation has been derived for transforming the coordinates of an observable block companion form to the coordinates of a block Schwarz form. The new method has partially extended the scalar Routh criterion to the matrix Routh criterion to a class of multivariable systems.

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